ALTERNATING BLOCK SUMS OF DECIMAL POWERS AND A CORRECTION TO A386609

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Abstract. Pieter Post's sequence A386609 asks for the least exponent $b \geq 2$ for which the absolute alternating sum of the two-digit decimal blocks of n^b equals n. At $n = 1000$ the table we audited lists $b = 3$. We prove that the correct entry is zero, since every exponent gives block sum 1 or 10. For blocks of any fixed width we classify the inputs that are powers of ten. An exact audit of the first 1001 entries in each of A386609 and A390112 finds one discrepant entry across the two tables.

1. The return equation

Let $B \geq 2$ and write $m = \sum_{j=0}^{q} d_j B^j$, with $0 \leq d_j < B$. Set

$$A_B(m) = \sum_{j=0}^{q} (-1)^j d_j, \qquad S_B(m) = |A_B(m)|, \qquad S_B(0) = 0.$$

For $B = 10^w$ the d_j are blocks of w decimal digits, read from the right. We write $a_w(n)$ for the least $b \geq 2$ such that $S_{10^w}(n^b) = n$, and set $a_w(n) = 0$ when there is no such exponent. Pieter Post's entries A386609 and A390112 give a_2 and a_3, respectively [1, 2]. Both entries suggest that the positive powers of the block base account for the inputs having no return. The test case 1000 at width two allows us to settle that suggestion directly.

Theorem 1. *For every integer $w \geq 1$ and every $e \geq 0$,*

$$S_{10^w}(10^e) = 10^{e \bmod w}.$$

Proof. Write $e = wq + r$ with $0 \leq r < w$. The identity $10^e = 10^r (10^w)^q$ places a single nonzero digit, 10^r, in position q of the base-10^w expansion. Its contribution to A_{10^w} is $(-1)^q 10^r$. Taking the absolute value gives the formula. □

Corollary 2. *For integers $w \geq 1$ and $t \geq 0$,*

$$a_w(10^t) = \begin{cases} 1 + w/\gcd(w,t), & 0 \leq t < w, \\ 0, & t \geq w. \end{cases}$$

In particular $a_2(1000) = 0$.

Proof. Apply Theorem 1 with $e = tb$. When $t < w$, a return is equivalent to $tb \equiv t \pmod{w}$. Thus $b - 1$ must be a positive multiple of $w/\gcd(w,t)$, giving the stated least exponent. When $t \geq w$, Theorem 1 gives $S_{10^w}((10^t)^b) \leq 10^{w-1} < 10^t$ for every b. □

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For $w = 2$ and $t = 3$, the block sum is 1 for even b and 10 for odd b. In particular the cube groups as $10 \mid 00 \mid 00 \mid 00 \mid 00$, so its block sum is 10. For each $w \geq 2$ the inputs 10^{wq+1}, $q \geq 1$, furnish infinitely many exceptions outside the powers of 10^w. At $w = 3$ the first such input is 10000. The argument at that input for A390112 was given earlier by `npnelson` [3].

2. An exact audit

For each index $0 \leq n \leq 1000$ in each table, we tested the listed value against the definition. At $n = 0$, $S_B(0) = 0$ gives the positive minimum $b = 2$. A positive claim $a_w(n) = b_0$ requires checking both the return at b_0 and the absence of an earlier return. From $B \equiv -1 \pmod{B+1}$ we obtain

$$m \equiv A_B(m) \pmod{B+1}, \qquad S_B(n^b) = n \implies n^b \equiv n \text{ or } -n \pmod{B+1}.$$

The audit scans every b from 2 through b_0, retaining the exponents passing this necessary congruence. For each survivor it computes the exact power with GMP and sums its blocks. Corollary 2 determines the values at the decimal powers at or above the block base, including the listed zero entries.

Table	Verified minima	Listed zeros confirmed	Discrepant indices
A386609	999	1	1000 (lists 3, correct 0)
A390112	1000	1	None

The C11 verifier records the SHA-256 digests of both tables retrieved on November 28, 2026. Its source and the exact download and build commands are available in the versioned audit program. The input files are linked in [1, 2].

3. Formal verification

The Lean proof uses Lean 4.33.1 and its standard library. For any valid digit list, `alternating_ofDigits` equates the quotient-and-remainder computation with the list's alternating sum. Theorem 1 is `decimal_power`. The cube is checked in `A386609_cube`, and the bound for every exponent is `A386609_no_exponent`. The file can be replayed with

```
lean -DwarningAsError=true proofs/oeis-a386609.lean
```

References

[1] P. Post, A386609, *The On-Line Encyclopedia of Integer Sequences*, 2025. Table of terms.
[2] P. Post, A390112, *The On-Line Encyclopedia of Integer Sequences*, 2025. Table of terms.
[3] N. P. Nelson (`npnelson`), Formal counterexample to the A390112 existence conjecture at $n = 10000$, Open Math Agents, pull request 46640.