KOTESOVEC'S WIDTH-SEVEN RECURRENCE CONJECTURE FOR BOUNDED-DISPLACEMENT DERANGEMENTS

COLE DEXTER AND GPT 6.2 ASTRA

Abstract. In 2015 V. Kotesovec predicted a recurrence of order 1780 for the number of derangements in which each displacement is at most seven in absolute value. We prove that 1780 is the least possible order over the rationals, even when a recurrence may start after an arbitrary finite prefix. A paired transfer gives a monic recurrence from index zero. A nonsingular Hankel window modulo 65521 and an invertible second transfer exclude every smaller order. Lean verifies both bounds, including the identification of the finite certificate with the permutation counts. Supplementary computations at widths eight and nine use external certificate programs and the same mathematical argument.

1. Statement of the result

For $k \geq 1$, let $a_k(n)$ count the permutations π of $\{1, \ldots, n\}$ such that $1 \leq |\pi(i)-i| \leq k$ for every i, and put $a_k(0) = 1$. This is the permanent of the $n \times n$ matrix whose (i, j) entry is 1 if $1 \leq |i - j| \leq k$ and 0 otherwise. A rational recurrence of order r with starting index N means

$$\sum_{j=0}^{r} c_j a_k(n + j) = 0 \quad (n \geq N), \qquad c_j \in \mathbb{Q}, \quad c_r \neq 0. \tag{1}$$

In a July 7, 2015 comment on Heinz's array [2], V. Kotesovec conjectured that column k satisfies a recurrence of order

$$d_k = \tfrac{1}{2}\left(\binom{2k}{k} + 2^k\right) \tag{2}$$

for each $k > 0$. The width-six entry records an order-494 recurrence. The entries for widths seven through nine list conjectural orders.

Theorem 1. *For $k = 7$, the least possible order in* (1) *is* 1780*. A monic recurrence of this order holds from $N = 0$. Every rational recurrence valid from any finite N has order at least* 1780*.*

The upper bound in (2) is part of the symmetric-band theory of Alekseyev and Khomovsky [1, Theorem 5.2]. We include a transfer proof to specify the integer upper-bound recurrence and the state system used in verification. The lower bound requires showing that the particular 0–1 specialization retains the full order.

Date: November 28, 2026.

Key words and phrases. Bounded-displacement permutations, Toeplitz permanents, linear recurrences, Hankel matrices.

Cole Dexter and GPT 6.2 Astra are the co-authors and owners of this work.

2. The paired transfer

Proposition 2. *For every $k \geq 1$, the sequence a_k satisfies a monic integer recurrence of order d_k from index zero.*

Proof. View a permutation as a perfect matching between row vertices and column vertices. Reveal row i and column i together. At a cut, keep the unmatched rows and columns among the last k positions as a pair of subsets (U, V) of a k-set. They have the same size because equally many rows and columns have been revealed and each edge uses one of each. The new row can match an available old column or stay unmatched. The new column can independently match an available old row or stay unmatched. The two new vertices cannot match each other because the diagonal is zero. Require that every vertex leaving the window has been matched. Then shift the window and record the new unmatched sets. Starting and ending with the empty state gives exactly the desired matchings, since each edge is recorded when its later endpoint appears.

Transposition swaps U and V and preserves every move multiplicity. For a source orbit, choose either representative and sum the multiplicities to all representatives of each target orbit. This gives a well-defined integer transfer matrix on unordered pairs. The empty state is fixed, so its return counts are unchanged by the quotient. There are $\sum_j \binom{k}{j}^2 = \binom{2k}{k}$ ordered states, of which 2^k are fixed. The quotient has dimension d_k. The monic characteristic polynomial supplies the recurrence by Cayley–Hamilton. □

3. An invertible row transfer

Proposition 3. *For $k > 0$, the sequence $a_k(n)$ is a boundary-to-boundary matrix coefficient of an integer transfer matrix of determinant ± 1.*

Proof. Before processing row i, store a bit for each occupied column in $[i-k, i+k-1]$, ordered with the oldest column first. In the initial state the low k bits represent the unavailable nonpositive columns. Append a free bit at the right. Mark a free column other than the central one, require the oldest bit to be marked, and discard it. A return to the initial bit pattern after n moves means all genuine columns up to n have been used and no larger one has been used. With the choices of columns retained as edge labels, this gives a bijection between closed boundary walks and the permutations counted by $a_k(n)$.

Consider all 2^{2k} masks. Sort sources by their low bit and destinations by their high bit. A source with low bit 0 must choose the oldest column, giving a bijection to destinations with high bit 0. A source with low bit 1 can choose the appended column, giving a bijection to destinations with high bit 1. In both bijections the source is recovered by reversing the shift and restoring its low bit. The remaining choices from a low-bit-1 source all lead to high-bit-0 destinations. Thus the two diagonal blocks are permutation matrices and one off-diagonal block vanishes. Taking the determinant gives the claim. □

For any prime p, the transfer acts bijectively on its finite vector space over $\mathbb{F}_p$. Its initial vector therefore returns to itself after a positive number of steps. In particular every finite Hankel window of the sequence modulo p reappears arbitrarily late. This lets an early certificate bound the order of a recurrence with unknown starting index.

4. Hankel certificates

Fix $p = 65521$. For a chosen width put $d = d_k$, $s_i = a_k(i+1) \bmod p$, and $S(X) = \sum_{i=0}^{2d-1} s_i X^i$.

Lemma 4. *Suppose $P, Q \in \mathbb{F}_p[X]$ satisfy*

$$QS \equiv P \pmod{X^{2d}}, \qquad \deg Q = d, \quad \deg P < d, \quad \gcd(P,Q) = 1. \tag{3}$$

The $d \times d$ Hankel matrix $H = (s_{i+j})_{0 \le i,j < d}$ is nonsingular.

Proof. A putative nonzero kernel vector $(v_0, \dots, v_{d-1})$ gives the nonzero polynomial $R = \sum_{j=0}^{d-1} v_j X^{d-1-j}$. For $0 \le i < d$, the coefficient of X^{d-1+i} in RS is the ith kernel equation. Hence $RS \equiv T \pmod{X^{2d-1}}$ for a polynomial T of degree at most $d-2$. Together with (3), this gives $RP \equiv QT \pmod{X^{2d-1}}$. Each side has degree at most $2d-2$, so $RP = QT$. Coprimality implies $Q \mid R$, impossible for nonzero R of degree less than d. □

Proof of Theorem 1. At width seven the 3,561-term prefix from $a_7(0)$ gives a certificate with $\deg P = 1779$, $\deg Q = 1780$ and $Q(0) = 1$. Lean verifies (3) by convolution and a Bézout identity, and identifies all input residues with the actual permutation counts [4]. Lemma 4 gives a nonsingular window of order 1780.

Given a recurrence with starting index N, Proposition 3 moves the same modular Hankel window to an index at least N. The integer matrix at that position has determinant nonzero modulo p, hence nonzero over $\mathbb{Q}$. If $r < d$, the vector $(c_0, \dots, c_r, 0, \dots, 0)$ in (1) would be a nonzero rational kernel vector of that matrix. This proves $r \ge d$ and Proposition 2 gives equality. □

Formal verification and reproducibility

The all-width upper bound and the interpretation of both transfers are proved in Lean. At width seven, Lean also checks the entire computation. It decodes the packed base-2^{64} polynomial coefficients, proves that integer convolution causes no cross-lane carries, verifies $X^{3560} \mid QS - P$, and checks a Bézout identity. For the packed transfer it verifies the effect of compaction, masked additions and modular reductions on the 2^{14} state coordinates. A symbolic predecessor proof for the fifteen column choices identifies the update with the row transfer. The 3,561 checked terms are thereby identified with the actual permutation counts modulo 65521.

The final declaration in [4] gives an order-1780 monic rational recurrence from zero and excludes all smaller eventual orders. A re-import in Lean 4.33.1 reports only `propext`, `Classical.choice` and `Quot.sound`. Direct elimination gives $\det H = 38307$ in $\mathbb{F}_{65521}$ at width seven. The generated residues agree with the published OEIS data through $n = 1000$ [2].

Supplementary computations at widths eight and nine

The two external C++ term generators in [3] implement the paired and row transfers. They agree on every term of the two prefixes whose lengths are given in Table 1. Direct convolution checks (3) and the Euclidean algorithm gives the nonzero constant remainders shown. The argument of Lemma 4 and the two transfer propositions therefore give minimal eventual rational orders 6563 and 24566, respectively. These lower bounds rely on the external computations and the written argument. The

generated prefixes agree with the OEIS data through $n = 1000$ at width eight and $n = 300$ at width nine [2].

TABLE 1. Supplementary finite-field computations at widths eight and nine. The prefix length includes $a_k(0)$. The final column gives the last nonzero Euclidean remainder.

k	Prefix length	$\deg P$	$\deg Q$	$[X^d]Q$	Remainder
8	13,127	6,562	6,563	65,520	48,484
9	49,133	24,565	24,566	65,520	19,044

Corollary 5. *At widths seven, eight and nine, the generic symmetric banded Toeplitz permanent also has minimal recurrence order d_k over its rational function field, both with a free diagonal and with a fixed zero diagonal.*

Proof. For each proposed starting index N, the preceding proof gives a Hankel window beginning at or after N whose determinant is nonzero at the 0–1 specialization. The determinant is a polynomial in the band parameters, so it is nonzero over the rational function field. The matching upper bound holds for both families by [1, Theorem 5.2]. □

REFERENCES

[1] M. A. Alekseyev and D. I. Khomovsky, *Symmetry reductions and recurrence degrees for banded Toeplitz determinants and permanents*, preprint (2026). arXiv:2609.23224.

[2] A. P. Heinz, square array of bounded-displacement derangement counts and its columns, with conjectures by V. Kotesovec, *The On-Line Encyclopedia of Integer Sequences* (2015). A259776, A259779, A259780, A259781, A259782.

[3] C. Dexter and GPT 6.2 Astra, *Bounded-displacement term generators, residue polynomials and certificate checks* (2026). Certificate files and reproduction instructions.

[4] C. Dexter and GPT 6.2 Astra, *Minimal rational recurrence order for width seven*, Lean formalization (2026). Final theorem and supporting modules.