MATCHINGS IN A TRIANGULAR PATCH OF THE HONEYCOMB LATTICE

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ABSTRACT. Let $L_{n,k}$ count tilings of the equilateral triangle of side n by k lozenges and $n^2 - 2k$ unit triangles. We show that, for $n \geq \lceil k/2 \rceil$, this count is a polynomial in n of degree $2k$ with leading coefficient $(3/2)^k/k!$. The threshold is sharp for every $k \geq 2$ when the empty board is included. An expansion of matching-polynomial deletion ratios in rooted ordered trees shows that the quadratic formula for the jth power sum first holds permanently at $n = \lceil j/2 \rceil$ for $j \geq 2$. An earlier independent Lean proof by NOELOO6 establishes eventual polynomiality with this degree and leading coefficient, and Mathar's five- and six-lozenge formulas. A separate Lean proof establishes both formulas for the matching counts on every side $n \geq 3$ by reducing local vertex-deletion calculations to 64 boundary configurations.

1. The fixed-column problem

A lozenge is the union of two adjacent cells in the subdivision of an equilateral triangle of integer side n into n^2 unit triangles. The boundary and the three edge directions are fixed. We write $L_{n,k}$ for the number of tilings with k lozenges. Mathar computed several columns and proposed formulas for five and six lozenges in Conjectures 1 and 2 of [3]. The OEIS records data for the full array [5]. We give a single stability argument for every fixed column.

NOELOO6's independent Lean proof establishes Mathar's two formulas for every $n \geq 3$, as well as eventual polynomiality of every fixed column with degree $2k$ and leading coefficient $(3/2)^k/k!$. It also establishes the resulting eventual identities for backward differences of orders $2k$ and $2k+1$ [4]. The public statement linked the signed proof on November 28, 2026. On November 29 the corresponding proof submission opened at 00:41:50 UTC, followed by our independent submission at 00:54:23 UTC. Theorem 1.1 determines the first stable side for every $k \geq 2$ and proves failure at the preceding side.

Theorem 1.1. *For each $k \geq 1$, there is a rational polynomial P_k of degree $2k$ and leading coefficient $(3/2)^k/k!$ such that*

$$L_{n,k} = P_k(n) \qquad (n \geq 1,\ n \geq \lceil k/2 \rceil).$$

With the empty board at side zero, $P_k(\lceil k/2 \rceil - 1) \neq L_{\lceil k/2 \rceil - 1,k}$ for every $k \geq 2$. For $k = 1$ the formula also holds at side zero.

Let G_n be the bipartite graph whose vertices are the unit triangles and whose edges join triangles sharing a side. A k-edge matching specifies the k lozenges uniquely. Give downward triangles the integer coordinates

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$$D_n = \{(x, y) : x, y \geq 0,\ x + y \leq n - 2\}.$$

The upward neighbors of $d \in D_n$ have coordinates $d + o$, with $o \in \{(0,0), (1,0), (0,1)\}$. Upward and downward triangles with the same coordinates are distinct. In particular, the board has $3\binom{n}{2}$ edges.

Write $Z_G(t) = \sum_j m_j(G)t^j$ for the matching generating polynomial of a finite graph G, where $m_j(G)$ counts the matchings of size j. Define the formal power series $R(G, v) = Z_{G-v}(-t)/Z_G(-t)$. Splitting according to the edge incident with v, if present, gives

$$R(G, v) = \left(1 - t \sum_{w \sim v} R(G - v, w)\right)^{-1}.$$

This vertex-deletion identity is a form of Godsil's path-tree method for matching polynomials [2]. We use its positive expansion to track exactly how much of the triangular boundary a given coefficient can see.

For $j \geq 1$ put $q_j(n) = \sum_{d \in D_n} [t^j]R(G_n, d)$. A matching of size i leaves $|D_n| - i$ downward vertices unoccupied. Hence $\sum_d Z_{G_n - d}(t) = |D_n| Z_{G_n}(t) - t Z'_{G_n}(t)$. Substitution of $-t$ and coefficient comparison give

$$kL_{n,k} = \sum_{j=1}^{k} (-1)^{j-1} q_j(n) L_{n,k-j}, \qquad L_{n,0} = 1. \tag{1}$$

2. Rooted trees and the triangular boundary

Lemma 2.1. *For $j \geq 1$, the function q_j agrees with a rational polynomial Q_j of degree at most two when $n \geq \lceil j/2 \rceil$. If $j \geq 2$, then $Q_j(n) > q_j(n)$ at $n = \lceil j/2 \rceil - 1$, with the graph at side zero taken to be empty.*

Proof. Expand the reciprocal in the deletion identity as a geometric series. A contribution to $[t^j]R(G, d)$ is a rooted ordered tree with j edges and a map of its vertices to G that takes its root to d and each edge to an edge. Vertices along any single root-to-leaf path have distinct images, because the recursion deletes each successive ancestor. Different branches can have the same images. Conversely, the ordered list of subtrees at the root recovers exactly one term of the expansion. Every such map contributes with weight one.

Work first on the infinite honeycomb lattice. Up to translation there are finitely many decorated trees of size j, since each is specified by the ordered tree and one of three edge directions at each step. Discard decorations with a repeated vertex on one branch. The surviving decorations and their multiplicities are independent of n. For one decoration let S be the set of its downward coordinates and put

$$b(S) = \max_{p \in S}(p_x + p_y) - \min_{p \in S} p_x - \min_{p \in S} p_y.$$

An integer translate fits when every downward coordinate is in D_n. Every upward coordinate is then on the board because it is adjacent to one of these downward coordinates. After shifting the two coordinate minima to zero, allowed translation vectors (u, v) satisfy $u, v \geq 0$ and $u + v \leq n - 2 - b(S)$. The number of translates is $\binom{n-b(S)}{2}$ when $n \geq b(S)$, including the two zero values at $n = b(S), b(S) + 1$. For smaller n there are no translates.

Choose downward vertices A, B, C of the abstract tree attaining respectively the minimum x, minimum y, and maximum $x + y$. The three pairwise paths in the tree share a unique vertex M, their median. The three arms from M to A, B, C have total length at

most j. If M is downward, the arms have even lengths. Each pair of edges can decrease x or y, or increase $x+y$, by at most one. Summing the three bounds for $M_x - A_x$, $M_y - B_y$, and $(C_x + C_y) - (M_x + M_y)$ gives $b(S) \le j/2$. If M is upward, the arms have odd lengths $2a+1, 2b+1, 2c+1$. Their first steps can decrease x and y by at most one and cannot increase $x+y$. The three bounds are now $a+1, b+1, c$, giving $b(S) \le (j+1)/2$. A repeated extremal vertex yields a downward median with a zero-length arm, already covered by the even case.

Thus $Q_j(n)$ is the sum over decorations of the polynomials $\binom{n-b(S)}{2}$, where $\binom{z}{2} = z(z-1)/2$ for all z. To see sharpness for $j = 2r$, a simple path starting downward and moving r times from one downward cell to the next in direction $(1,0)$ has span r. For $j = 2r+1 \ge 3$, take the three downward neighbors $(-1,0), (0,-1), (0,0)$ of the upward origin. Root the three-edge tree at $(-1,0)$ and extend its $(0,0)$ branch by $r-1$ positive horizontal jumps between downward cells. The span is $r+1$. At one side length below maximal span, every maximal decoration contributes $\binom{-1}{2} = 1$ to the polynomial and zero to the graph count. All smaller decorations agree there. Positivity gives strict inequality. □

Proof of Theorem 1.1. Apply (1) inductively in k. The term $q_1(n)L_{n,k-1}$ has degree $2k$ since $q_1(n) = 3\binom{n}{2}$. Every $j \ge 2$ contributes degree at most $2 + 2(k-j) \le 2k-2$. This proves the degree and leading coefficient in Theorem 1.1. At $n = (k-1)/2$ for odd $k \ge 3$, $q_j(n) = Q_j(n)$ for $j < k$, and $L_{n,i} = P_i(n)$ for $1 \le i < k$. Only q_k differs, and its coefficient in (1) is $+1$. The strict discrepancy in Lemma 2.1 proves odd-column sharpness. Section 3 proves the remaining assertion. □

3. Sharpness in the even columns

Let $A_{j,b}$ be the number of surviving rooted decorations of span b in Lemma 2.1. Write $r = \lceil j/2 \rceil$. For $j \ge 2$, its proof gives $Q_j(r-1) - q_j(r-1) = A_{j,r}$. Two such discrepancies interact in an even matching column. We first count the odd one exactly.

Lemma 3.1. *For $r \ge 2$, put $U_r = A_{2r-1,r}/(2r-1)$ and $V_r = A_{2r,r}/(2r)$. Then*

$$U_r = 2^{r-1}\binom{r}{2}, \qquad V_r \le 6(2r-1)U_r + 2^{r+1}\binom{r+2}{2}.$$

Proof. For an unrooted decorated tree T with j edges let $W(T) = \prod_v (\deg v - 1)!$. Fixing a downward vertex v as root gives at most $\deg(v)W(T)$ orders of the children, by permuting the children independently at each vertex. Since the downward degrees sum to j, the total is at most $jW(T)$. For an embedded tree, distinct roots and orders give distinct admissible decorations. Indeed, a translation preserving a finite nonempty image must be the identity, and an embedding identifies each vertex by its image. The bound is therefore attained.

Suppose $j = 2r-1$ and the span is r. In the median proof of Lemma 2.1 the median must be upward. Equality forces the three arms to use every edge and attain their individual displacement bounds. Translate the median to the origin. On the arm attaining minimum x, every upward-to-downward step decreases x, and every reverse-color step preserves x with two choices. On the arm attaining minimum y the corresponding assertions hold for y. On the third arm every upward-to-downward step preserves the coordinate sum, and every reverse-color step increases it with two choices. Outside the median these arms lie, respectively, in $x < 0$, in $x \ge 0, y < 0$, and in $x, y \ge 0$. The first

two arms have negative coordinate sum at every downward vertex, while the third has nonnegative sum there. Thus each selected extremum is unique among the downward vertices. The three arms are simple because the chosen projection advances strictly at every second step. Consequently each construction gives a distinct embedded tree.

Writing the three lengths as $2u+1, 2v+1, 2w+1$ gives $u+v+w = r-2$. There are $\binom{r}{2}$ length triples and 2^{r-2} decorations for each. Only the median has degree three, so $W = 2$. This gives the value of U_r.

Now let $j = 2r$. For an upward median the three odd arm lengths must total $2r-1$, and all three displacement bounds are attained. The tree is therefore one of the preceding embedded trees with one additional pendant edge. At an attachment vertex v there are at most three image directions, and W increases by the factor $\deg(v)$. The degrees in the core sum to $2(2r-1)$. Summing the resulting upper bounds for rooted orders gives a contribution to V_r of at most $6(2r-1)U_r$.

For a downward median the even lengths must total $2r$. Write them as $2u, 2v, 2w$ with $u+v+w = r$, allowing zero lengths and coincident extrema. Each pair of steps realizing its extremal displacement has two choices: on the first two arms the downward-to-upward step preserves the chosen coordinate, and on the third it increases the coordinate sum. Thus there are at most $2^r\binom{r+2}{2}$ decorated trees, each with $W \le 2$. Counting images which repeat along a branch or trees with multiple choices of extremal vertices only increases this bound. □

Let $\Delta_r = P_{2r}(r-1) - L_{r-1,2r}$. The formal logarithm of the matching generating polynomial has coefficients $[t^j]\log Z_{G_n}(t) = (-1)^{j-1}q_j(n)/j$, by (1). Replacing q_j by Q_j gives the same relation for the formal series with coefficients P_j. At side $r-1$, for $r \ge 2$, these two logarithms agree through degree $2r-2$. At degrees $2r-1$ and $2r$ their differences are U_r and $-V_r$. Exponentiating through degree $2r$ gives

$$\Delta_r = 3\binom{r-1}{2}U_r - V_r.$$

For $r \ge 11$, Lemma 3.1 gives

$$\frac{V_r}{U_r} \le 6(2r-1) + \frac{4(r+2)(r+1)}{r(r-1)} \le 12r < 3\binom{r-1}{2}.$$

Hence $\Delta_r > 0$ for every such r.

For completeness the remaining discrepancies, including the empty board at $r = 1$, are

r	1	2	3	4	5
Δ_r	−3	−30	−208	−900	−2820
r	6	7	8	9	10
Δ_r	−6548	−9012	11532	148748	738828.

Here is a finite recurrence for reproducing this table. For each side $n \le 12$, process downward triangles in lexicographic order. For each set T of occupied upward triangles adjacent to a remaining downward triangle store a polynomial $B_T(t)$ through degree 20, initially $B_\varnothing = 1$. At the next downward triangle d, add B_T to the state T and add tB_T to the state $T \cup \{u\}$ for each unoccupied upward neighbor u of d. Intersect each destination with the upward neighbors of the unprocessed triangles and combine identical states. Every matching follows a unique sequence of these choices. The final polynomial is $Z_{G_n}(t)$ through the required degree. Compute $q_j(n)$ by (1); for $1 \le j \le 2r$, recover $Q_j(r-1) = 3q_j(r) - 3q_j(r+1) + q_j(r+2)$ by quadratic extrapolation. Applying (1)

again gives the displayed differences. The accompanying Lean certificate evaluates the ten numbers using a diagonal matching transfer. A separate implementation evaluates the vertex-deletion recurrence directly and gives the same numbers.

4. Five and six lozenges

Corollary 4.1. *For $n \geq 3$ put $M = 3\binom{n}{2}$. Then*

$$\begin{aligned} L_{n,5} &= \frac{3(n-3)(n+3)}{1280}\big(27n^8 - 135n^7 - 387n^6 + 2835n^5 \\ &\qquad - 168n^4 - 18732n^3 + 19568n^2 + 36992n - 56320\big), \\ L_{n,6} &= \binom{M}{6} - 3(n-1)^2\binom{M-2}{4} \\ &\quad + \frac{1}{16}\big(153n^8 - 918n^7 - 90n^6 + 9828n^5 - 10851n^4 \\ &\qquad - 41206n^3 + 69420n^2 + 61472n - 131088\big). \end{aligned}$$

Proof. All six q_j have stabilized by side three, so three sides determine their quadratics. The deletion recurrence gives

n	q_1	q_2	q_3	q_4	q_5	q_6
3	9	33	135	585	2619	11961
4	18	72	330	1632	8478	45546
5	30	126	612	3222	17880	102948

Writing $(-1)^{j-1}Q_j(n) = a_j + b_j n + c_j\binom{n}{2}$, the rows (a_j, b_j, c_j) for $j = 1, \ldots, 6$ are

$$\begin{pmatrix} 0 & 0 & 3 \\ -6 & 6 & -15 \\ 72 & -66 & 87 \\ -702 & 582 & -543 \\ 6300 & -4770 & 3543 \\ -54108 & 37866 & -23817 \end{pmatrix}.$$

Substitution in (1), beginning with $L_{n,0} = 1$, gives the asserted formulas. The expressions agree with [3, eqs. (123)–(124)]. In the expanded version of (124) the sign preceding $19678080n$ is missing; expanding its binomial form gives a plus sign. □

The six-lozenge formula also holds at side three, one side below the conjectured range. Its polynomial evaluates to -208 at side two, where no six-lozenge tiling exists. Thus side three is the exact first stable side for both formulas. The previously solved cases with two, three, and four lozenges are credited to Greg Dresden and Eldin Sijaric in the corresponding OEIS entries [5].

5. Positive coefficients and verification

With $A_{j,b}$ as above, put $h(m) = \binom{m}{2}$ for integers $m \geq 2$, and put $h(m) = 0$ otherwise. The proof of Lemma 2.1 gives $q_j(n) = \sum_b A_{j,b} h(n-b)$ for every $n \geq 0$. Consequently,

$$\sum_{n \geq 0} q_j(n) x^n = \frac{x^2 \sum_b A_{j,b} x^b}{(1-x)^3}.$$

Since $h(2) = 1$, the value at side $b + 2$ determines $A_{j,b}$ from the preceding entries. At side two the sole downward vertex has three upward leaves, giving $q_j(2) = 3^j$. Using the three sides in Corollary 4.1 gives the following certificate valid for all sides.

j	$A_{j,0}$	$A_{j,1}$	$A_{j,2}$	$A_{j,3}$
1	3	0	0	0
2	9	6	0	0
3	27	54	6	0
4	81	342	120	0
5	243	1890	1350	60
6	729	9774	11850	1464

There is a second proof of Corollary 4.1 that avoids the enumeration of abstract trees. Write the barycentric coordinates of a downward triangle as (a, b, c), where $a+b+c = n-2$. Through degree six, the coefficients of its deletion ratio depend only on $\min(a, 3)$, $\min(b, 3)$ and $\min(c, 3)$. Along a walk of at most six edges from the root, a barycentric coordinate can decrease by one only on an upward-to-downward step. There are at most three such steps. The proof tracks the deleted ancestors along the walk, so it also preserves the exclusions in the recurrence.

For these 4^3 coordinate triples the seven coefficients from degree zero to six can be evaluated directly. The resulting function is a sum of 20 terms supported on thresholds $a \geq i$, $b \geq j$, $c \geq \ell$. A threshold admits $h(n - i - j - \ell)$ roots, by subtracting its three lower bounds. Grouping by $i + j + \ell$ gives the table above. Applying (1) to the actual board matchings now proves the two formulas for all $n \geq 3$.

Lean verifies this second proof and the bijection that fills every unmatched triangle with a singleton tile. Its final theorem counts tilings in which each board triangle is covered exactly once [1]. The argument for arbitrary k is the rooted-tree proof of Theorem 1.1. Two independent programs, using vertex deletion and a diagonal matching transfer, also agree on all 126 values for $1 \leq n \leq 18$ and $0 \leq k \leq 6$.

References

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