A DIMENSION CLASSIFICATION FOR MENG'S MULTIPLIER BOUND AND THE FOUR-DIMENSIONAL CUBE-FREE BOUND

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ABSTRACT. Yuchen Meng asked whether a subset of $\mathbb{Z}_N$ omitting at least one of $x, 2x, \ldots, (d-1)x$ for every nonzero x has size at most $(d-1)N/d$ whenever $d \mid N$. We prove that the bound holds for precisely the dimensions $2 \le d \le 4$. For $d = 4$, a 3-adic weight gives $4|A| \le 3N$ for every modulus when the multiplier condition includes zero. Rounding gives the nonzero-generator bound when $4 \mid N$. Meng's reduction gives $4|A| \le 3N$ for every projective 4-cube-free set at every modulus. For every $d \ge 5$, finite cyclic versions of Chapman's integer valuation construction give counterexamples on infinitely many moduli. In July 2026, hypnopump had already refuted the multiplier conjecture at $(5, 40)$, lifted that example to every $(5, 40q)$, and exhibited examples for $d = 7, 8$. We give the dimension classification, formal proofs in Lean, and a separate finite computation for the smallest counterexample modulus and the extremal problems at $(5, 40)$.

1. Introduction and Main Results

Sum-free subsets of integers and finite abelian groups—sets avoiding solutions to $x + y = z$—date back to Schur [12] and remain a central theme in additive combinatorics, alongside their generalizations to (k, l)-sum-free sets [1, 7, 8, 9, 13]. A natural higher-order generalization is the study of *projective d-cube-free* subsets. Throughout this paper, let $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ denote the cyclic group of order $N \ge 1$, and write $[d] := \{1, 2, \ldots, d\}$. Meng's Definitions 1–2 specify projective cubes. For multiplier patterns we distinguish the two choices of generator used below [11, §4.1].

Definition 1.1 (Projective d-cubes and multiple-free subsets)**.** Let $N \ge 1$ and $d \ge 2$ be integers, and let $A \subseteq \mathbb{Z}_N$.

(i) Given a multiset $S = \{a_1, \ldots, a_d\}$ of d (not necessarily distinct) elements of $\mathbb{Z}_N$, the *projective d-cube* generated by S is the set of all nonempty subset sums

$$\Sigma^* S := \left\{ \sum_{i \in I} a_i : \emptyset \ne I \subseteq [d] \right\} \subseteq \mathbb{Z}_N.$$

We say that A is *d-cube-free* if $\Sigma^* S \not\subseteq A$ for every multiset S of size d in $\mathbb{Z}_N$.

(ii) For $x \in \mathbb{Z}_N$, put $M_d(x) = \{x, 2x, \ldots, (d-1)x\}$. Call a set *all-generator multiple-free* if $M_d(x) \not\subseteq A$ for every $x \in \mathbb{Z}_N$, and *nonzero-generator multiple-free* if this is required for every $x \ne 0$. The unqualified pattern-free condition in the proofs below means the all-generator condition.

The all-generator condition forces $0 \notin A$ since $M_d(0) = \{0\}$. Projective cube-freeness also forces $0 \notin A$, by taking every generator zero. Meng explicitly uses the family indexed by $x \in \mathbb{Z}_N \setminus \{0\}$ in Section 4.1. A set satisfying that condition may contain zero.

Motivated by a conjecture of Eberhard and Pohoata [11, Conjecture 3] for 3-cube-free subsets of $[N]$ and earlier work of Long and Wagner [10] on $\mathbb{Z}_{2^n}$, Meng [11] formulated two conjectures in the cyclic group $\mathbb{Z}_N$:

- **Conjecture 4 ([11]):** *If $A \subseteq \mathbb{Z}_N$ is d-cube-free and $d \mid N$, then $|A| \le \frac{d-1}{d} N$.*

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- **Conjecture 10 ([11]):** *If $A \subseteq \mathbb{Z}_N$ is nonzero-generator multiple-free and $d \mid N$, then $|A| \leq \frac{d-1}{d}N$.*

Meng [11, Theorem 6] proved that the bound $\frac{d-1}{d}N$ in Conjecture 4 is tight whenever $d \mid N$ via $A = \{a \in \mathbb{Z}_N : a \not\equiv 0 \pmod{d}\}$. Crucially, Meng [11, Section 3] proved that *Conjecture 10 implies Conjecture 4*: if $A \subseteq \mathbb{Z}_N$ is d-cube-free and fails to be $\{x, 2x, \ldots, (d-1)x\}$-free, then picking x with $\{x, \ldots, (d-1)x\} \subseteq A$ and considering the d-cubes $\{x, \ldots, x, y\}$ for all $y \in \mathbb{Z}_N$ forces $A \cap (A-x) \cap \cdots \cap (A-(d-1)x) = \emptyset$, hence $|A| \leq \frac{d-1}{d}N$. Using this strategy, Meng [11, Theorems 5, 9, and Section 4] proved both conjectures when: (i) $d = 3$, (ii) d is the smallest prime factor of N, or (iii) $N = p^\ell$ is a prime power. Comparing Conjecture 10 with Corollary 14 of [11] (which shows that $\{x, (d-1)x\}$-free subsets have density at most $\frac{k}{k+1} \leq \frac{d-1}{d}$ where $k = \gcd(d-1, N)$), Meng noted that one might expect forbidding $\{x, 2x, \ldots, (d-1)x\}$ to impose the same maximum density $\frac{d-1}{d}$.

The IIIS Lean project formalized Meng's diagonal reduction, three positive regimes and sharp examples [5]. On July 26, 2026, hypnopump published a Lean refutation of Conjecture 10 at $(5, 40)$, gave the infinite family $(5, 40q)$ and the cube in Theorem 1.6, and supplied further counterexamples at $(7, 63)$ and $(8, 48)$ [4]. The present paper determines every dimension in which Conjecture 10 holds and settles the four-dimensional case of Conjecture 4 for every modulus.

Theorem 1.2 (Complete Dichotomy for Meng's Conjecture 10)**.** *Let $d \geq 2$ be an integer. The statement*

$$\forall N \geq 1 \text{ with } d \mid N,\ \forall A \subseteq \mathbb{Z}_N \text{ nonzero-generator multiple-free}, \quad d|A| \leq (d-1)N$$

holds if and only if $d \leq 4$. *The same classification holds for the all-generator condition.*

The estimate used for 4-cube-free sets has the following form for all moduli.

Theorem 1.3 (All-generator multiplier bound for $d = 4$)**.** *For every integer $N \geq 1$, every all-generator multiple-free set $A \subseteq \mathbb{Z}_N$ for $d = 4$ satisfies $4|A| \leq 3N$.*

Theorem 1.4 (Unconditional Resolution of Conjecture 4 for $d = 4$)**.** *For every integer $N \geq 1$* (in particular whenever $4 \mid N$)*, every 4-cube-free subset $A \subseteq \mathbb{Z}_N$ satisfies*

$$4|A| \leq 3N.$$

On the negative side, Conjecture 10 fails dramatically for every $d \geq 5$: not only does a counterexample exist for each $d \geq 5$, but infinitely many moduli divisible by d exhibit a constant positive density excess above $\frac{d-1}{d}$.

Theorem 1.5 (Universal Refutation of Conjecture 10 for All $d \geq 5$)**.** *For every integer $d \geq 5$, there exist an explicit modulus $N_0 \geq 1$ with $d \mid N_0$ and an explicit $\{x, 2x, \ldots, (d-1)x\}$-free subset $A_0 \subseteq \mathbb{Z}_{N_0}$ satisfying*

$$d|A_0| > (d-1)N_0.$$

Moreover, for every integer $t \geq 1$, the preimage $A_t := \pi^{-1}(A_0) \subseteq \mathbb{Z}_{tN_0}$ under the canonical ring homomorphism $\pi : \mathbb{Z}_{tN_0} \to \mathbb{Z}_{N_0}$ is $\{x, 2x, \ldots, (d-1)x\}$-free, satisfies $d \mid tN_0$, and obeys

$$d|A_t| \geq (d-1)(tN_0) + t.$$

The following earlier example separates the two forbidden configurations [4].

Theorem 1.6 (The 33-point witness of hypnopump at $(5, 40)$)**.** *In $\mathbb{Z}_{40}$, the subset*

$$A_{40} := \mathbb{Z}_{40} \setminus \{0, 4, 12, 20, 24, 28, 36\}$$

is $\{x, 2x, 3x, 4x\}$-free and satisfies $|A_{40}| = 33 > 32 = \frac{4}{5} \cdot 40$. However, A_{40} is not 5-cube-free: the multiset $S = \{1, 1, 1, 5, 8\}$ satisfies

$$\Sigma^* S = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16\} \subseteq A_{40}.$$

The July search identified 40 as the smallest counterexample modulus. Section 5 explains the exact finite checks for both choices of generator and the separate maximum of 32 for 5-cube-free sets at $N = 40$. The general projective cube question for $d \geq 5$ remains open.

Remark 1.7 (Compatibility with Meng's Theorem 5). In [11, Section 4], Meng established Conjecture 10 when d is the smallest prime factor of N or when $N = p^\ell$ is a prime power. Every counterexample produced by Theorems 1.5 and 1.6 has a composite modulus divisible by a prime strictly smaller than d ($40 = 2^3 \cdot 5$, $N = 2^{2m}d$ in Case A, and $N = 3 \cdot 2^{4r}d$ in Case B), matching the exact boundary of Meng's positive regimes.

2. Counting Preliminaries in $\mathbb{Z}_N$

For each $x \in \mathbb{Z}_N$, we denote by $\tilde{x} \in \{0, 1, \ldots, N-1\}$ its unique canonical representative. We first record two elementary counting lemmas in $\mathbb{Z}_N$, together with the base cases $d = 2$ and $d = 3$.

Lemma 2.1 (Fiber cardinality bound). *For every $N \geq 1$, every $c \geq 1$, and every $y \in \mathbb{Z}_N$,*

$$\big|\{x \in \mathbb{Z}_N : cx = y\}\big| \leq c.$$

Proof. Let $F := \{x \in \mathbb{Z}_N : cx = y\}$. If $F = \emptyset$, the bound is trivial. Otherwise, fix $x_0 \in F$. For any $x \in F$, we have $c(x - x_0) = 0$ in $\mathbb{Z}_N$, so $N \mid c(\widetilde{x - x_0})$. Since $0 \leq (\widetilde{x - x_0}) < N$, the integer $q(x) := c(\widetilde{x - x_0})/N$ satisfies $0 \leq q(x) < c$ and $c(\widetilde{x - x_0}) = Nq(x)$. If $q(x_1) = q(x_2)$ for $x_1, x_2 \in F$, then $c(x_1 - x_0) = c(x_2 - x_0)$, so $(x_1 - x_0) = (x_2 - x_0)$ and hence $x_1 = x_2$ in $\mathbb{Z}_N$. Thus $x \mapsto q(x)$ injects F into $\{0, 1, \ldots, c-1\}$, giving $|F| \leq c$. □

Lemma 2.2 (Divisibility and layer counts). *Let $N \geq 1$.*

(i) *If $m \geq 1$ and $m \mid N$, then $\big|\{x \in \mathbb{Z}_N : m \mid \tilde{x}\}\big| = N/m$.*
(ii) *If $2^{j+1} \mid N$ for $j \geq 0$, then $\big|\{x \in \mathbb{Z}_N : 2^j \mid \tilde{x} \text{ and } 2^{j+1} \nmid \tilde{x}\}\big| = N/2^{j+1}$.*
(iii) *If $3 \cdot 2^{j+1} \mid N$ for $j \geq 0$, then*

$$\big|\{x \in \mathbb{Z}_N : 3 \mid \tilde{x},\ 2^j \mid \tilde{x},\ \text{and } 2^{j+1} \nmid \tilde{x}\}\big| = \frac{N}{3 \cdot 2^{j+1}}.$$

Proof. For (i), the map $k \mapsto mk \bmod N$ bijects $\{0, 1, \ldots, N/m - 1\}$ onto $\{x \in \mathbb{Z}_N : m \mid \tilde{x}\}$, because $m \cdot (N/m) = N$. For (ii), $\{x : 2^{j+1} \mid \tilde{x}\} \subseteq \{x : 2^j \mid \tilde{x}\}$, so by (i) the difference has size $N/2^j - N/2^{j+1} = N/2^{j+1}$. For (iii), since $\gcd(3, 2^j) = \gcd(3, 2^{j+1}) = 1$, we have $(3 \mid \tilde{x} \wedge 2^j \mid \tilde{x}) \iff 3 \cdot 2^j \mid \tilde{x}$ and similarly for 2^{j+1}, so subtracting the two sets via (i) gives $N/(3 \cdot 2^j) - N/(3 \cdot 2^{j+1}) = N/(3 \cdot 2^{j+1})$. □

Proposition 2.3 (Base cases $d = 2$ and $d = 3$). *For every integer $N \geq 1$:*

(i) *Every $\{x\}$-free subset $A \subseteq \mathbb{Z}_N$ is empty, so $2|A| = 0 \leq N$.*
(ii) *Every $\{x, 2x\}$-free subset $A \subseteq \mathbb{Z}_N$ satisfies $3|A| \leq 2N$.*

Proof. Part (i) is immediate since $\{x\}$-freeness means $x \notin A$ for every $x \in \mathbb{Z}_N$. For (ii) (cf. [11, Theorem 9]), let $B := \mathbb{Z}_N \setminus A$. For every $x \in \mathbb{Z}_N$, at least one of x or $2x$ lies in B, so $1 \leq \mathbf{1}_B(x) + \mathbf{1}_B(2x)$. Summing over all $x \in \mathbb{Z}_N$ and grouping the second sum by $y = 2x \in B$ yields, via Lemma 2.1 with $c = 2$,

$$N \leq |B| + \sum_{y \in B} \big|\{x \in \mathbb{Z}_N : 2x = y\}\big| \leq |B| + 2|B| = 3|B|.$$

Since $|A| + |B| = N$, this gives $3|A| = 3(N - |B|) \leq 2N$. □

3. Affirmative Proof for $d = 4$ via a 3-Adic Dual Weight Function

Why does the naive uniform counting proof of Proposition 2.3(ii) fail for $d = 4$? In $\mathbb{Z}_N$, a naive uniform sum over $1 \leq \mathbf{1}_B(x) + \mathbf{1}_B(2x) + \mathbf{1}_B(3x)$ gives at most $1 + 2 + 3 = 6$ hits per element of B, yielding only $|B| \geq N/6$ instead of the sharp $N/4$. The key insight behind Theorem 1.3 is that multiplication by 2 preserves 3-adic valuation, whereas multiplication by 3 shifts 3-adic valuation

upward by 1. By assigning alternating weights 4 and 0 to even and odd 3-adic layers below $b = v_3(N)$ (and carefully calibrating the boundary layer $v = b$ to weight 3 when b is even and 1 when b is odd), we construct an exact linear-programming dual witness certifying $|B| \geq N/4$ for *every* $N \geq 1$.

3.1. **The Truncated 3-Adic Valuation and Weight Function.** Fix $N \geq 1$ and let $b \coloneqq v_3(N) \geq 0$ be the 3-adic valuation of N, so $3^b \mid N$ and $3^{b+1} \nmid N$. For any integer $n \geq 0$, define the *truncated 3-adic valuation* $v_b(n) \in \{0, 1, \ldots, b\}$ recursively by

$$v_0(n) \coloneqq 0, \qquad v_{b+1}(n) \coloneqq \begin{cases} 1 + v_b(n/3) & \text{if } 3 \mid n, \\ 0 & \text{if } 3 \nmid n. \end{cases}$$

By induction on $k \leq b$, we have $k \leq v_b(n) \iff 3^k \mid n$, and in particular $v_b(n) = b \iff 3^b \mid n$. Three immediate properties follow from this characterization:

(V1) *Well-definedness modulo N:* Since $3^b \mid N$, we have $3^k \mid (n \bmod N) \iff 3^k \mid n$ for all $k \leq b$, so $v_b(n \bmod N) = v_b(n)$. We may therefore write $v(x) \coloneqq v_b(\tilde{x})$ for $x \in \mathbb{Z}_N$.

(V2) *Invariance under doubling:* Since $\gcd(3^k, 2) = 1$, we have $v_b(2n) = v_b(n)$, hence $v(2x) = v(x)$ for all $x \in \mathbb{Z}_N$.

(V3) *Shift under tripling:* For all $n \geq 0$, $v_b(3n) = \min(b, v_b(n) + 1)$, hence $v(3x) = \min(b, v(x) + 1)$ for all $x \in \mathbb{Z}_N$.

Definition 3.1 (The 3-adic dual weight function)**.** For $b \geq 0$ and $n \geq 0$, define $w_b(n) \in \{0, 1, 3, 4\}$ by

$$w_b(n) \coloneqq \begin{cases} 4 & \text{if } v_b(n) < b \text{ and } v_b(n) \text{ is even}, \\ 0 & \text{if } v_b(n) < b \text{ and } v_b(n) \text{ is odd}, \\ 3 & \text{if } v_b(n) = b \text{ and } b \text{ is even}, \\ 1 & \text{if } v_b(n) = b \text{ and } b \text{ is odd}. \end{cases} \tag{3.1}$$

For $x \in \mathbb{Z}_N$ (with $b = v_3(N)$), we write $w(x) \coloneqq w_b(\tilde{x})$.

Note immediately that $w_b(n) \leq 4$ always, $w_0(n) = 3$, $w_1(n) = 1$ if $3 \mid n$ and 4 otherwise, and for every $b \geq 0$ we have the period-2 recurrence

$$w_{b+2}(n) = \begin{cases} 4 & \text{if } 3 \nmid n, \\ 0 & \text{if } 3 \mid n \text{ and } 9 \nmid n, \\ w_b(n/9) & \text{if } 9 \mid n. \end{cases} \tag{3.2}$$

Indeed, if $9 \mid n$, then $v_{b+2}(n) = 2 + v_b(n/9)$, which preserves both the inequality $v < b$ (versus $v + 2 < b + 2$) and the parity of v and b.

3.2. **Total Weight Identity and Uniform Fiber Bound.**

Lemma 3.2 (Total weight identity)**.** *For every $b \geq 0$ and every $N \geq 1$ with $3^b \mid N$,*

$$\sum_{x \in \mathbb{Z}_N} w_b(\tilde{x}) = 3N.$$

Proof. We proceed by strong induction on b.

- *Base case $b = 0$:* $w_0(\tilde{x}) = 3$ for all $x \in \mathbb{Z}_N$, so the sum is $3N$.
- *Base case $b = 1$:* Here $3 \mid N$. By Lemma 2.2(i), exactly $N/3$ elements of $\mathbb{Z}_N$ satisfy $3 \mid \tilde{x}$ (each of weight 1), and the remaining $N - N/3 = 2N/3$ elements satisfy $3 \nmid \tilde{x}$ (each of weight 4). The total weight is
$$\frac{N}{3} \cdot 1 + \frac{2N}{3} \cdot 4 = 3N.$$
- *Inductive step $b + 2$:* Suppose $3^{b+2} \mid N$. Then $9 \mid N$ and $3^b \mid (N/9)$. Using (3.2), partition $\mathbb{Z}_N$ into three sets:
  (a) $\{x \in \mathbb{Z}_N : 3 \nmid \tilde{x}\}$ has cardinality $2N/3$ and constant weight 4, contributing $8N/3$;

(b) $\{x \in \mathbb{Z}_N : 3 \mid \tilde{x},\ 9 \nmid \tilde{x}\}$ has constant weight 0, contributing 0;
(c) $\{x \in \mathbb{Z}_N : 9 \mid \tilde{x}\}$ bijects onto $\mathbb{Z}_{N/9}$ via $y \mapsto 9\tilde{y} \bmod N$, sending $w_{b+2}(\tilde{x}) = w_b(\tilde{x}/9)$ to $w_b(\tilde{y})$. By the inductive hypothesis for b and $N/9$, this part contributes $3(N/9) = N/3$.

Summing the three contributions gives $8N/3 + 0 + N/3 = 3N$. □

Lemma 3.3 (Uniform fiber hit bound)**.** *Let $N \geq 1$ and $b = v_3(N)$ (so $3^b \mid N$ and $3^{b+1} \nmid N$). For every $y \in \mathbb{Z}_N$,*

$$w(y) + \sum_{\substack{x \in \mathbb{Z}_N \\ 2x = y}} w(x) + \sum_{\substack{x \in \mathbb{Z}_N \\ 3x = y}} w(x) \leq 12. \tag{3.3}$$

Proof. By property (V2), every $x \in \mathbb{Z}_N$ with $2x = y$ satisfies $v(x) = v(y)$ and hence $w(x) = w(y)$. By Lemma 2.1 with $c = 2$, there are at most 2 such x, so

$$w(y) + \sum_{2x = y} w(x) \leq 3w(y).$$

Let $F_3 \coloneqq \{x \in \mathbb{Z}_N : 3x = y\}$. By Lemma 2.1 with $c = 3$, $|F_3| \leq 3$, and by property (V3), every $x \in F_3$ satisfies

$$\min(b, v(x) + 1) = v(y). \tag{3.4}$$

It suffices to prove that $3w(y) + \sum_{x \in F_3} w(x) \leq 12$. Write $N = 3^b m$ with $3 \nmid m$. We analyze the four exhaustive cases for $v(y) \in \{0, \ldots, b\}$.

Case 1: $v(y) < b$ and $v(y)$ is even. Here $w(y) = 4$. For any $x \in F_3$, (3.4) gives $v(x) + 1 = v(y) < b$. (In particular, if $v(y) = 0$, no such x can exist and $F_3 = \emptyset$.) Thus $v(x) = v(y) - 1 < b$ is odd, so $w(x) = 0$ for every $x \in F_3$. Hence $3w(y) + \sum_{x \in F_3} w(x) = 3 \cdot 4 + 0 = 12$.

Case 2: $v(y) < b$ and $v(y)$ is odd. Here $w(y) = 0$. Since $w(x) \leq 4$ for every x and $|F_3| \leq 3$, we have

$$3w(y) + \sum_{x \in F_3} w(x) \leq 0 + 4|F_3| \leq 12.$$

Case 3: $v(y) = b$ and b is even. Here $w(y) = 3$. We claim that the subset $F_{\text{top}} \coloneqq \{x \in F_3 : 3^b \mid \tilde{x}\}$ has cardinality at most 1. Indeed, suppose $x_1, x_2 \in F_{\text{top}}$. Then $3(x_1 - x_2) = 0$ in $\mathbb{Z}_N$, so $N = 3^b m \mid 3(x_1 - x_2)$. Since $3 \nmid m$, $\gcd(m, 3) = 1$, forcing $m \mid (x_1 - x_2)$. On the other hand, $3^b \mid \tilde{x}_1$ and $3^b \mid \tilde{x}_2$ imply $3^b \mid (x_1 - x_2)$ (as $3^b \mid N$). Since $\gcd(3^b, m) = 1$, it follows that $N = 3^b m \mid (x_1 - x_2)$, whence $(x_1 - x_2) = 0$ and $x_1 = x_2$.

Thus $|F_{\text{top}}| \leq 1$, and every $x \in F_{\text{top}}$ has $v(x) = b$ and weight $w(x) = 3$. For every $x \in F_3 \setminus F_{\text{top}}$, we have $v(x) < b$; by (3.4), $\min(b, v(x) + 1) = b$, forcing $v(x) = b - 1$. Since b is even, $b - 1$ is odd, so $w(x) = 0$ for all $x \in F_3 \setminus F_{\text{top}}$. Consequently,

$$3w(y) + \sum_{x \in F_3} w(x) = 9 + \sum_{x \in F_{\text{top}}} 3 + 0 \leq 9 + 3 = 12.$$

Case 4: $v(y) = b$ and b is odd. Here $b \geq 1$ and $w(y) = 1$. If $|F_3| \leq 2$, then $3w(y) + \sum_{x \in F_3} w(x) \leq 3 + 4|F_3| \leq 11 \leq 12$. Assume therefore that $|F_3| = 3$ (so $F_3 \neq \emptyset$) and choose $x_0 \in F_3$. By (3.4), $v(x_0) \geq b - 1$, so $\tilde{x}_0 = 3^{b-1} q$ for some $q \in \mathbb{N}$. Also $3 \mid N$ and $N/3 = 3^{b-1} m$ with $3 \nmid m$. Because $m \equiv 1$ or $2 \pmod 3$, there exists $t \in \{0, 1, 2\}$ such that $3 \mid (q + tm)$. Define

$$x_{\text{top}} \coloneqq x_0 + t(N/3) \in \mathbb{Z}_N.$$

Then $3x_{\text{top}} = 3x_0 + tN = y$ in $\mathbb{Z}_N$, so $x_{\text{top}} \in F_3$. Moreover, $\tilde{x}_0 + t(N/3) = 3^{b-1}(q + tm)$ is divisible by 3^b, and since $3^b \mid N$, we have $3^b \mid \tilde{x}_{\text{top}}$, i.e., $v(x_{\text{top}}) = b$, whence $w(x_{\text{top}}) = 1$. Each of the remaining $|F_3 \setminus \{x_{\text{top}}\}| = 2$ elements has weight at most 4, so

$$3w(y) + \sum_{x \in F_3} w(x) = 3 + w(x_{\text{top}}) + \sum_{x \in F_3 \setminus \{x_{\text{top}}\}} w(x) \leq 3 + 1 + 2 \cdot 4 = 12.$$ □

3.3. **Proofs of Theorems 1.3 and 1.4.**

Proof of Theorem 1.3. Let $A \subseteq \mathbb{Z}_N$ be $\{x, 2x, 3x\}$-free, and put $B \coloneqq \mathbb{Z}_N \setminus A$, so $|A| + |B| = N$. For every $x \in \mathbb{Z}_N$, at least one of $x, 2x, 3x$ belongs to B, so the non-negative weight $w(x)$ satisfies

$$w(x) \leq w(x)\mathbf{1}_B(x) + w(x)\mathbf{1}_B(2x) + w(x)\mathbf{1}_B(3x).$$

Summing over all $x \in \mathbb{Z}_N$, applying Lemma 3.2, grouping the three sums fiberwise by $y \in B$, and invoking Lemma 3.3 yields

$$3N = \sum_{x \in \mathbb{Z}_N} w(x) \leq \sum_{y \in B} \left(w(y) + \sum_{\substack{x \in \mathbb{Z}_N \\ 2x = y}} w(x) + \sum_{\substack{x \in \mathbb{Z}_N \\ 3x = y}} w(x) \right) \leq \sum_{y \in B} 12 = 12|B|.$$

Dividing by 3 gives $N \leq 4|B|$, hence $4|A| = 4(N - |B|) \leq 3N$. □

Corollary 3.4 (The nonzero-generator condition)**.** *If $2 \leq d \leq 4$, $d \mid N$, and $A \subseteq \mathbb{Z}_N$ is nonzero-generator multiple-free, then $d|A| \leq (d-1)N$.*

Proof. Put $B = \mathbb{Z}_N \setminus A$. For $d = 4$, sum the weighted inequality above over $x \neq 0$. The total weight is $3N - w(0) \geq 3N - 4$, and restricting the sum can only decrease the weight received by each $y \in B$. Thus $12|B| \geq 3N - 4$. Write $N = 4m$. If $|B| \leq m - 1$, then $12|B| \leq 3N - 12$, a contradiction. For $d = 3$, omit zero in the unweighted count in Proposition 2.3. It gives $3|B| \geq N - 1$, hence $|B| \geq N/3$ when $3 \mid N$. For $d = 2$ one has $A \subseteq \{0\}$, and $2 \mid N$, $N \geq 1$ imply $2|A| \leq 2 \leq N$. □

Proof of Theorem 1.4. Let $A \subseteq \mathbb{Z}_N$ be 4-cube-free. If A is $\{x, 2x, 3x\}$-free, then $4|A| \leq 3N$ by Theorem 1.3. Otherwise, there exists $x \in \mathbb{Z}_N$ such that $x, 2x, 3x \in A$. Put $B \coloneqq \mathbb{Z}_N \setminus A$. We claim that for every $y \in \mathbb{Z}_N$, at least one of $y, y + x, y + 2x, y + 3x$ lies in B. Indeed, if all four belonged to A, then the multiset $S = \{x, x, x, y\}$ of size 4 would have nonempty subset sums

$$\Sigma^* S = \{x, 2x, 3x, y, y + x, y + 2x, y + 3x\} \subseteq A,$$

contradicting the 4-cube-freeness of A. Therefore, for every $y \in \mathbb{Z}_N$,

$$1 \leq \mathbf{1}_B(y) + \mathbf{1}_B(y + x) + \mathbf{1}_B(y + 2x) + \mathbf{1}_B(y + 3x).$$

Summing over all $y \in \mathbb{Z}_N$ and using translation invariance of counting in $\mathbb{Z}_N$ gives $N \leq 4|B|$, whence $4|A| = 4(N - |B|) \leq 3N$. □

4. Universal Refutation of Conjecture 10 for Every $d \geq 5$

We now prove Theorems 1.6, 1.5, and 1.2. The core mechanism behind every counterexample is that in a composite cyclic group $\mathbb{Z}_N$, a sparse excluded set $B = \mathbb{Z}_N \setminus A$ concentrated on specific 2-adic (or $\{2, 3\}$-adic) valuation layers can intercept every orbit $\{x, 2x, \ldots, (d-1)x\}$ using multipliers $k \in \{1, \ldots, d-1\}$ that are powers of 2 (or 3 times a power of 2), while keeping $|B| < N/d$.

4.1. **The July Counterexample at $(d, N) = (5, 40)$.**

Proof of Theorem 1.6. Let $B_{40} \coloneqq \{0, 4, 12, 20, 24, 28, 36\} \subseteq \mathbb{Z}_{40}$ and $A_{40} \coloneqq \mathbb{Z}_{40} \setminus B_{40}$. Clearly $5 \mid 40$, $|B_{40}| = 7$, and $|A_{40}| = 40 - 7 = 33 > 32 = \frac{4}{5} \cdot 40$.

To verify that A_{40} is $\{x, 2x, 3x, 4x\}$-free, let $x \in \mathbb{Z}_{40}$ and observe that B_{40} contains the entire layer $E \coloneqq \{4, 12, 20, 28, 36\} = \{z \in \mathbb{Z}_{40} : 4 \mid \tilde{z} \text{ and } 8 \nmid \tilde{z}\}$ together with $\{0, 24\} \subseteq 8\mathbb{Z}_{40}$:

- If $\tilde{x}$ is odd ($v_2(\tilde{x}) = 0$), then $4x \in E \subseteq B_{40}$.
- If $2 \mid \tilde{x}$ and $4 \nmid \tilde{x}$ ($v_2(\tilde{x}) = 1$), then $2x \in E \subseteq B_{40}$.
- If $4 \mid \tilde{x}$ and $8 \nmid \tilde{x}$ ($v_2(\tilde{x}) = 2$), then $1 \cdot x \in E \subseteq B_{40}$.
- If $8 \mid \tilde{x}$, then $x \in \{0, 8, 16, 24, 32\}$. We check directly in $\mathbb{Z}_{40}$:

$$1 \cdot 0 = 0, \qquad 3 \cdot 8 = 24, \qquad 4 \cdot 16 = 64 \equiv 24, \qquad 1 \cdot 24 = 24, \qquad 2 \cdot 32 = 64 \equiv 24,$$

all of which lie in $\{0, 24\} \subseteq B_{40}$.

Thus for every $x \in \mathbb{Z}_{40}$, some $k \in \{1,2,3,4\}$ satisfies $kx \in B_{40}$, so A_{40} is $\{x, 2x, 3x, 4x\}$-free.

Finally, for the multiset $S = \{1,1,1,5,8\}$ in $\mathbb{Z}_{40}$, every nonempty subset sum has the form $a+5b+8c$ with $a \in \{0,1,2,3\}$, $b,c \in \{0,1\}$, $(a,b,c) \neq (0,0,0)$. Enumerating all combinations yields

$$\Sigma^* S = \{1,2,3,5,6,7,8,9,10,11,13,14,15,16\},$$

which is disjoint from $B_{40} = \{0,4,12,20,24,28,36\}$; hence $\Sigma^* S \subseteq A_{40}$ and A_{40} is not 5-cube-free. □

4.2. **Universal Construction for All $d \geq 5$.** Given any integer $d \geq 5$, set $m \coloneqq \lfloor \log_2(d-1) \rfloor + 1$. Then $m \geq 3$ and $2^{m-1} < d \leq 2^m$. We divide into two exhaustive cases: Case A ($2^{m-1} < d \leq 2^m - 2$) and Case B ($d \in \{2^m - 1, 2^m\}$).

Note that whenever $m_0 \mid N$ and $k \in \mathbb{N}$, we have $m_0 \mid \widetilde{kx} \iff m_0 \mid k\tilde{x}$ for every $x \in \mathbb{Z}_N$ (because $\widetilde{kx} = (k\tilde{x}) \bmod N$). Also, if $2^L \nmid \tilde{x}$, then there exists a unique $v \in \{0,1,\ldots,L-1\}$ such that $2^v \mid \tilde{x}$ and $2^{v+1} \nmid \tilde{x}$.

Chapman's integer construction supplies the valuation pattern for both cases [2, Lemma 3.3]. For $a,k \geq 2$, let $S(a,k)$ contain the positive integers with $\nu_a(n) \equiv k-1 \pmod{k}$, where $\nu_a(n) = \max\{j : a^j \mid n\}$. Every set $n\{1,a,\ldots,a^{k-1}\}$ meets $S(a,k)$, which has natural density $(a-1)/(a^k-1)$. In Case A, $S(2,m)$ has density $1/(2^m-1) < 1/d$. We retain its first valuation layer and include every residue divisible by 2^{2m-1}, making a periodic blocker that also covers zero. In Case B, $3S(2,r)$ meets every $n\{3, 3\cdot 2, \ldots, 3\cdot 2^{r-1}\}$. Its density is $1/[3(2^r-1)] < 1/d$ for $d = 2^{r+1}-1$ or 2^{r+1} and $2^r \geq 4$. We retain its first three valuation layers and include the terminal multiples of 2^{4r-1}. The propositions below verify that these finite cyclic cutoffs preserve the strict density inequalities and give moduli divisible by d.

Proposition 4.1 (Case A: $2^{m-1} < d \leq 2^m - 2$)**.** *Let $m \geq 3$ and $2^{m-1} < d \leq 2^m - 2$. Set $N \coloneqq 2^{2m}d$, define $B \coloneqq B_1 \cup B_2 \subseteq \mathbb{Z}_N$ where*

$$B_1 \coloneqq \{x \in \mathbb{Z}_N : 2^{m-1} \mid \tilde{x} \text{ and } 2^m \nmid \tilde{x}\}, \qquad B_2 \coloneqq \{x \in \mathbb{Z}_N : 2^{2m-1} \mid \tilde{x}\},$$

and put $A \coloneqq \mathbb{Z}_N \setminus B$. Then $d \mid N$, A is $\{x, 2x, \ldots, (d-1)x\}$-free, and $d|A| > (d-1)N$.

Proof. Clearly $d \mid N$ and $2^{2m} \mid N$. To prove that A is $\{x,\ldots,(d-1)x\}$-free, let $x \in \mathbb{Z}_N$. If $2^{2m-1} \mid \tilde{x}$, then $1 \cdot x \in B_2 \subseteq B$ (and $1 < d$). Otherwise $2^{2m-1} \nmid \tilde{x}$, so there exists $v \in \{0,\ldots,2m-2\}$ with $2^v \mid \tilde{x}$ and $2^{v+1} \nmid \tilde{x}$:

- If $v \leq m-1$, put $k \coloneqq 2^{m-1-v}$. Then $1 \leq k \leq 2^{m-1} < d$. Since $2^{m-1} = k \cdot 2^v \mid k\tilde{x}$ while $2^m = k \cdot 2^{v+1} \nmid k\tilde{x}$ (and $2^m \mid N$), we have $kx \in B_1 \subseteq B$.
- If $m \leq v \leq 2m-2$, put $k \coloneqq 2^{2m-1-v}$. Then $1 \leq k \leq 2^{m-1} < d$. Since $2^{2m-1} = k \cdot 2^v \mid k\tilde{x}$ (and $2^{2m-1} \mid N$), we have $kx \in B_2 \subseteq B$.

Hence A is $\{x,\ldots,(d-1)x\}$-free.

Now we compute $|B|$. Since $m \leq 2m-1$, B_1 and B_2 are disjoint. By Lemma 2.2(i)–(ii),

$$|B_1| = \frac{N}{2^m} = 2^m d, \qquad |B_2| = \frac{N}{2^{2m-1}} = 2d,$$

so $|B| = (2^m+2)d$. Because $d \leq 2^m - 2$, we obtain

$$|B| = (2^m+2)d \leq (2^m+2)(2^m-2) = 2^{2m} - 4 < 2^{2m} = \frac{N}{d}.$$

Multiplying by d yields $d|B| < N$, and therefore $d|A| = d(N - |B|) > (d-1)N$. □

Notice why Case A stops at $d = 2^m - 2$: for $d \in \{2^m-1, 2^m\}$, we have $(2^m+2)d > 2^{2m}$, so a pure 2-adic two-layer set B is slightly too dense. In Case B, introducing an auxiliary factor of 3 reduces the layer densities by a factor of 3 while increasing the largest required multiplier by only $3/2$, which fits comfortably below $2^m - 1$ for all $m \geq 3$.

Proposition 4.2 (Case B: $d \in \{2^m-1, 2^m\}$)**.** *Let $m \geq 3$ and $2^m - 1 \leq d \leq 2^m$. Set $r := m-1 \geq 2$, $R := 2^r \geq 4$, and $N := 3 \cdot 2^{4r} d = 3R^4 d$. Define $B := L_1 \cup L_2 \cup L_3 \cup L_4 \subseteq \mathbb{Z}_N$, where*

$$L_q := \{x \in \mathbb{Z}_N : 3 \mid \tilde{x},\ 2^{qr-1} \mid \tilde{x},\ \text{and } 2^{qr} \nmid \tilde{x}\} \qquad (q \in \{1,2,3\}),$$
$$L_4 := \{x \in \mathbb{Z}_N : 2^{4r-1} \mid \tilde{x}\},$$

and put $A := \mathbb{Z}_N \setminus B$. Then $d \mid N$, A is $\{x, 2x, \ldots, (d-1)x\}$-free, and $d|A| > (d-1)N$.

Proof. Observe first that $2^m = 2R = 4 \cdot 2^{r-1}$ and $2^{r-1} \geq 2$, so

$$3 \cdot 2^{r-1} = 4 \cdot 2^{r-1} - 2^{r-1} \leq 2^m - 2 < 2^m - 1 \leq d.$$

Clearly $d \mid N$, $3 \mid N$, and $2^{4r} \mid N$. To verify $\{x, \ldots, (d-1)x\}$-freeness of A, let $x \in \mathbb{Z}_N$. If $2^{4r-1} \mid \tilde{x}$, then $1 \cdot x \in L_4 \subseteq B$. Otherwise $2^{4r-1} \nmid \tilde{x}$, so there exists $v \in \{0, \ldots, 4r-2\}$ with $2^v \mid \tilde{x}$ and $2^{v+1} \nmid \tilde{x}$:

- If $(q-1)r \leq v \leq qr-1$ for some $q \in \{1,2,3\}$, put $s := qr-1-v$, so $0 \leq s \leq r-1$. Take $k := 2^s$ if $3 \mid \tilde{x}$, and $k := 3 \cdot 2^s$ if $3 \nmid \tilde{x}$. In either case $1 \leq k \leq 3 \cdot 2^{r-1} < d$, $3 \mid k\tilde{x}$, and since $\gcd(3,2) = 1$ we have $2^{qr-1} = 2^s \cdot 2^v \mid k\tilde{x}$ while $2^{qr} = 2^s \cdot 2^{v+1} \nmid k\tilde{x}$. Because $3 \mid N$ and $2^{qr} \mid N$, this gives $kx \in L_q \subseteq B$.
- If $3r \leq v \leq 4r-2$, put $k := 2^{4r-1-v}$. Then $1 \leq k \leq 2^{r-1} < d$ and $2^{4r-1} = k \cdot 2^v \mid k\tilde{x}$ (with $2^{4r-1} \mid N$), so $kx \in L_4 \subseteq B$.

Thus A is $\{x, \ldots, (d-1)x\}$-free.

By Lemma 2.2(i) and (iii),

$$|L_1| = \frac{N}{3 \cdot 2^r} = R^3 d, \qquad |L_2| = \frac{N}{3 \cdot 2^{2r}} = R^2 d, \qquad |L_3| = \frac{N}{3 \cdot 2^{3r}} = Rd, \qquad |L_4| = \frac{N}{2^{4r-1}} = 6d.$$

(Indeed L_1, L_2, L_3, L_4 are pairwise disjoint, so equality holds; the union bound already suffices.) Since $d \leq 2^m = 2R$ and $R \geq 4$, we obtain

$$|B| \leq (R^3 + R^2 + R + 6)d \leq 2R(R^3 + R^2 + R + 6) = 2R^4 + 2R^3 + 2R^2 + 12R.$$

Now

$$3R^4 - (2R^4 + 2R^3 + 2R^2 + 12R) = R(R(R(R-2)-2)-12) \geq 4(4(4 \cdot 2 - 2) - 12) = 48 > 0,$$

so $|B| < 3R^4 = N/d$. Multiplying by d gives $d|B| < N$, whence $d|A| = d(N - |B|) > (d-1)N$. □

4.3. Pullback Lifting and Proofs of Theorems 1.5 and 1.2.

Lemma 4.3 (Pullback lifting lemma)**.** *Let $N, t \geq 1$ and $d \geq 2$, and let $\pi : \mathbb{Z}_{tN} \to \mathbb{Z}_N$ be the canonical projection $x \mapsto x \bmod N$. If $A \subseteq \mathbb{Z}_N$ is all-generator multiple-free, then $A_t := \pi^{-1}(A) \subseteq \mathbb{Z}_{tN}$ is $\{x, 2x, \ldots, (d-1)x\}$-free and satisfies $|A_t| = t|A|$.*

Proof. For every $x \in \mathbb{Z}_{tN}$, since A is $\{x, \ldots, (d-1)x\}$-free, there exists $k \in \{1, \ldots, d-1\}$ with $k\pi(x) \notin A$. Because π is a ring homomorphism, $\pi(kx) = k\pi(x) \notin A$, so $kx \notin A_t$. Thus A_t is $\{x, \ldots, (d-1)x\}$-free.

Moreover, for every $y \in A$, the fiber $\{x \in \mathbb{Z}_{tN} : \pi(x) = y\}$ is in bijection with $\{0, 1, \ldots, t-1\}$ via $q \mapsto (\tilde{y} + qN) \bmod tN$, so every fiber has cardinality exactly t. Summing over $y \in A$ gives $|A_t| = t|A|$. □

Proof of Theorem 1.5. Given $d \geq 5$, set $m := \lfloor \log_2(d-1) \rfloor + 1 \geq 3$, so $2^{m-1} < d \leq 2^m$. If $d \leq 2^m - 2$, Proposition 4.1 provides $N_0 \geq 1$ with $d \mid N_0$ and a $\{x, \ldots, (d-1)x\}$-free $A_0 \subseteq \mathbb{Z}_{N_0}$ with $d|A_0| > (d-1)N_0$; otherwise $d \in \{2^m - 1, 2^m\}$ and Proposition 4.2 provides (N_0, A_0).

In particular, $d|A_0| \geq (d-1)N_0 + 1$. For any integer $t \geq 1$, Lemma 4.3 shows that $A_t = \pi^{-1}(A_0) \subseteq \mathbb{Z}_{tN_0}$ is $\{x, \ldots, (d-1)x\}$-free with $|A_t| = t|A_0|$, while $d \mid N_0 \mid tN_0$ and

$$d|A_t| = t \cdot d|A_0| \geq t((d-1)N_0 + 1) = (d-1)(tN_0) + t.$$ □

Proof of Theorem 1.2. For nonzero generators and $2 \le d \le 4$, apply Corollary 3.4. For all generators, $2 \le d \le 4$ implies that $d|A| \le (d-1)N$ holds for every $N \ge 1$ and every $\{x, \dots, (d-1)x\}$-free $A \subseteq \mathbb{Z}_N$ by Proposition 2.3 (for $d = 2, 3$) and Theorem 1.3 (for $d = 4$). Conversely, if $d \ge 5$, Theorem 1.5 constructs $N_0 \ge 1$ with $d \mid N_0$ and a $\{x, \dots, (d-1)x\}$-free $A_0 \subseteq \mathbb{Z}_{N_0}$ violating $d|A_0| \le (d-1)N_0$. □

5. Small Counterexamples and Structural Separation from Conjecture 4

The first two parameter pairs in Table 1 occur in hypnopump's July report [4], which also gives $(7, 63)$ with $55 > 54$. The other four rows illustrate Propositions 4.1 and 4.2.

Table 1. Explicit counterexamples to Meng's Conjecture 10 ($d \mid N$, $|A| > \frac{d-1}{d}N$).

(d, N)	$\lvert B\rvert$	Excluded set $B = \mathbb{Z}_N \setminus A$	$\lvert A\rvert$	$\frac{d-1}{d}N$	Excess	Density $\lvert B\rvert/N$
$(5, 40)$	7	$\{0, 4, 12, 20, 24, 28, 36\}$	33	32	+5	$7/40 < 1/5$
$(8, 48)$	5	$\{0, 6, 28, 30, 44\}$	43	42	+8	$5/48 < 1/8$
$(5, 320)$	50	Case A ($m = 3$): $B_1 \cup B_2$	270	256	+70	$5/32 < 1/5$
$(6, 384)$	60	Case A ($m = 3$): $B_1 \cup B_2$	324	320	+24	$5/32 < 1/6$
$(7, 5376)$	630	Case B ($m = 3, r = 2$): $\bigcup_{q=1}^{4} L_q$	4746	4608	+966	$15/128 < 1/7$
$(8, 6144)$	720	Case B ($m = 3, r = 2$): $\bigcup_{q=1}^{4} L_q$	5424	5376	+384	$15/128 < 1/8$

Smallest modulus. A blocker is a complement B hitting each required $M_d(x)$. For each $d \mid N$, a counterexample exists exactly when there is a blocker of size at most $N/d - 1$. The July Z3 search used Meng's proved cases and reported the first failure at $(5, 40)$ [4]. The original version of this paper reported an integer linear programming check using CBC. The accompanying source now gives a direct exhaustive test by branching on a still-unhit pattern and trying each of its possible blockers. It finds no violating set among the 111 pairs $2 \le d \le N < 40$ with $d \mid N$, and only $d = 5$ at $N = 40$, under either convention.

The extremal problems at $(5, 40)$. For nonzero generators the July enumeration found four minimum blockers, each of size six, giving maximum $|A| = 34$ [4]. For all generators a blocker must contain zero. The four blockers of size at most seven consist of $\{0, 4, 12, 20, 28, 36\}$ together with one member of $\{8, 16, 24, 32\}$. The maximum all-generator multiple-free set therefore has size 33. The 33-point witness and its $40q$ pullbacks satisfy both conditions.

For the cube question, suppose $|A| \ge 33$ and put $B = \mathbb{Z}_{40} \setminus A$. If $x, 2x, 3x, 4x \in A$ for some x, at most $5|B| \le 35$ residues y fail one of the conditions $y + jx \in A$ for $0 \le j \le 4$. Any remaining y completes the projective cube (x, x, x, x, y). Otherwise B is one of the four seven-point blockers. The finite verifier checks an explicit five-generator cube in each complement. The 32 residues outside $5\mathbb{Z}_{40}$ form a 5-cube-free set by Meng's Theorem 6, giving maximum 32. The small-modulus exclusions and the enumeration of four blockers are exhaustive computation. The cube displayed in Theorem 1.6 also has a separate Lean proof.

6. Formal Verification in Lean 4

Table 2 identifies the principal declarations in the original file `proofs/MengCubeFree.lean`, using Lean 4.33.1 [3] and Mathlib revision `0df444a360eaa60ab8c11dca51a86af692955474`. Its multiplier predicate quantifies over all x. In particular, `counterexample40_spec` proves the size, freeness and violation for the named witness.

The companion `proofs/MengCubeFreeSupplement.lean` proves the nonzero-generator classification as `meng_conjecture_10_nonzero_resolution` in the same namespace. Its proof pulls A back to $\mathbb{Z}_{2N}$ and erases zero. For $d \ge 3$ a generator in the kernel of the projection has double zero, and

any other generator is handled by its image. The resulting set is all-generator multiple-free and has size at least $2|A| - 1$. Applying the all-generator bound in dimensions three and four and rounding when $d \mid N$ gives the stated classification. The companion also verifies the 14 distinct nonempty sums and their containment in A_{40} in namespace `MengReplay`, through `nonempty_sums` and `counterexample40_contains_cube`. The finite optimality results are checked separately by the exhaustive search in `proofs/meng-cube-free/verify_finite.py` [6].

TABLE 2. Correspondence between paper results and verified Lean 4 declarations in `proofs/MengCubeFree.lean`.

Paper Result	**Lean 4 Declaration (`MengCubeFree.*`)**
Definition 1.1	`IsMultipleFree`, `IsCubeFree`
Lemma 2.1	`card_filter_mul_eq_le`
Lemma 2.2	`card_filter_dvd_val`, `card_filter_exact_pow2`, `card_filter_three_exact_pow2`
Proposition 2.3	`card_le_one_half_of_multipleFree_two`, `card_le_two_thirds_of_multipleFree_three`
Definition 3.1	`val3`, `weight3`
Lemma 3.2	`sum_weight3_eq_three_mul`
Lemma 3.3	`weight3_hit_bound`
Theorem 1.3	`card_le_three_fourths_of_multipleFree_four`
Theorem 1.4	`card_le_three_fourths_of_cubeFree_four`
33-point multiplier witness	`counterexample40`, `counterexample40_spec`
Proposition 4.1	`refutation_case_A`
Proposition 4.2	`refutation_case_B`
Lemma 4.3	`pullback_multipleFree`
Theorem 1.5	`disproof_meng_conjecture_10_all_d_ge_5`, `disproof_meng_conjecture_10_infinite_family`
All-generator dichotomy	`meng_conjecture_10_complete_resolution`

AUTHOR CONTRIBUTIONS

Cole Dexter and Gemini 4 Neon (Antigravity) developed the principal results in the original paper, including the 3-adic weight, the finite cyclic constructions in all dimensions $d \geq 5$, and the original Lean proof. GPT 6.2 Astra (Codex agents) traced the July counterexample and the valuation antecedents, reconciled the two generator conventions, contributed the companion Lean proofs and independent finite checks, and revised the attribution and verification statements. Cole Dexter, Gemini 4 Neon and GPT 6.2 Astra jointly own this revised manuscript.

REFERENCES

[1] J. Bourgain, *Estimates related to sumfree subsets of sets of integers*, Israel J. Math. **97** (1997), 71–92.
[2] J. Chapman, *Partition regularity and multiplicatively syndetic sets*, Acta Arith. **196** (2020), no. 2, 109–138. doi:10.4064/aa190421-11-3.
[3] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in Automated Deduction – CADE 28, Lecture Notes in Comput. Sci. **12699**, Springer, Cham, 2021, pp. 625–635.
[4] hypnopump, *A counterexample to Conjecture 10 of Meng, On Cube-Free Problems*, July 26, 2026. Versioned report, Lean proof, search programs and data.
[5] IIIS Lean, *Cube-Free Problems in Lean*, September 2026. Formalization of Meng's proved cases.

[6] C. Dexter and Gemini 4 Neon, *Complete Resolution of Meng's Conjecture 10 and the $d = 4$ Case of Meng's Conjecture 4*, Lean 4 proof (`proofs/MengCubeFree.lean`) and verification note (`notes/meng-cube-free.md`), Original Lean source (2026).
[7] S. Eberhard, *Følner sequences and sum-free sets*, Bull. Lond. Math. Soc. **47** (2015), no. 1, 21–28.
[8] S. Eberhard, B. Green, and F. Manners, *Sets of integers with no large sum-free subset*, Ann. of Math. (2) **180** (2014), no. 2, 621–652.
[9] Y. Jing and S. Wu, *The largest (k, l)-sum-free subsets*, Trans. Amer. Math. Soc. **374** (2021), no. 7, 5163–5189.
[10] J. Long and A. Z. Wagner, *The largest projective cube-free subsets of $\mathbb{Z}_{2^n}$*, European J. Combin. **81** (2019), 156–171.
[11] Y. Meng, *On cube-free problems*, Electron. J. Combin. **33** (2026), no. 1, Paper No. 1.16, 11 pp., doi:10.37236/14052, arXiv:2311.12318.
[12] I. Schur, *Über die Kongruenz $x^m + y^m \equiv z^m \pmod p$*, Jahresber. Dtsch. Math.-Ver. **25** (1916), 114–117.
[13] T. Tao and V. H. Vu, *Additive Combinatorics*, Cambridge Stud. Adv. Math., vol. 105, Cambridge University Press, Cambridge, 2006.

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