PERFECT RECTANGULAR PACKIT! GAMES WITH HALF THE TURNS EXPANDED

COLE DEXTER AND GPT 6.2 ASTRA

Abstract. On turn t of *PackIt!*, a player places a rectangle of area t or $t+1$ on a rectangular grid. We prove that every rectangular board of area $2s(s+1)$, with s a positive integer, has a perfect packing in $2s$ turns. These are exactly the areas for which a perfect packing requires half the turns to use the larger area. We realize this by expanding the odd turns. The construction partitions two copies of the integers $1, \dots, s$ into equal-sum rows and then doubles one dimension of the tiling. It answers a question of Garrison, Heule and Subercaseaux and has been verified in Lean.

1. The half-expansion question

On a positive $m \times n$ grid, a move places an axis-aligned rectangle of positive integer side lengths. Its area on turn t is either t or $t+1$. Rectangles must lie inside the board and have disjoint interiors. A game is *perfect* when the rectangles cover every cell. A turn using area $t+1$ is an *expansion turn*.

Write $A = mn$ and $T_j = j(j+1)/2$. In a perfect game with j turns the area lies between T_j and $T_j + j = T_{j+1} - 1$. These intervals partition the nonnegative integers. Thus the unique K satisfying $T_K \le A < T_{K+1}$ is the required number of turns and the gap $\gamma = A - T_K$ is the required number of expansions. Garrison, Heule and Subercaseaux ask whether $2\gamma = K$ suffices for a perfect packing on every rectangle [2, §5, question (2)].

Theorem 1. *If $2\gamma = K$, the $m \times n$ board admits a perfect game. Its expansion turns can be taken to be exactly the odd turns.*

Put $s = \gamma$. Then $K = 2s$ and

$$A = T_{2s} + s = 2s(s+1).$$

Since $A > 0$, we have $s > 0$. Choosing expansions on the odd turns prescribes the areas $2, 2, 4, 4, \dots, 2s, 2s$. The main issue is fitting rectangles of these areas into every board of area $2s(s+1)$.

2. A partition into rows

Lemma 2. *Let s, k, L be positive integers with $kL = s(s+1)$ and $L \ge s$. The multiset containing two copies of each integer $1, \dots, s$ can be partitioned into k rows, each of sum L.*

Proof. For an integer interval $[a, b]$, match each i in its first copy to $a + b - i$ in its second. This gives $b - a + 1$ pairs of sum $a + b$. We use strong induction on s.

Case $L = s$. Use the two copies of s as singleton rows. The doubled interval $[1, s-1]$ supplies $s-1$ pairs of sum s, including zero pairs when $s = 1$. This gives $s + 1 = k$ rows.

Case $s < L \le 2s$. Put $r = L - s - 1$, so $0 \le r < s$. Pair the doubled interval $[r+1, s]$. Its endpoints sum to L, and it supplies $s - r$ rows. The remaining entries are two copies of $[1, r]$ and have total

$$r(r+1) = \big(k - (s-r)\big)L.$$

If $r = 0$, the rows already exhaust the multiset. Otherwise the residual row count is positive, and $L \ge r$, so induction supplies the rest.

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Cole Dexter and GPT 6.2 Astra are the co-authors and owners of this work.

Case $L > 2s$. The area equation gives $k < (s+1)/2$, hence $2k \le s$. Set $r = s-k$. The doubled interval $[r+1, s]$ gives k pairs of sum $P = r+1+s$. The remaining entries have total $r(r+1) = k(L-P)$. Since $r \ge k > 0$, the new width $L - P = r(r+1)/k$ is at least r. Induction partitions these entries into k rows of width $L - P$. Append one pair to each row to reach width L. □

There is also a short deduction of Lemma 2 from a theorem of Ando, Gervacio and Kano [1, Theorem 1]. For $k = 1$ take a single row. Otherwise put $h = \lfloor k/2 \rfloor$. Their theorem gives h disjoint subsets of $\{1, \dots, s\}$, each of sum L, whenever $2h - 1 \le L$ and $hL \le s(s+1)/2$. Here $2h - 1 \le k - 1 \le s \le L$, since $kL = s(s+1)$ and $L \ge s$. The second inequality follows from $2h \le k$. Use the h subsets in each copy of the interval to form $2h$ rows. The remaining entries have total $(k - 2h)L$. For even k this leaves no entries, and for odd k they form the last row.

3. A tiling in chronological order

Proof of Theorem 1. At least one side of the board is even. Halve an even side and orient the result as k rows of length L, with $k \le L$. This half-board has area $s(s+1)$. If $L < s$, its area would be strictly smaller than s^2, so $L \ge s$. Lemma 2 supplies the bar lengths for the k rows. Lay the bars end to end in each row. Every cell of the half-board is covered once.

Double the dimension previously halved, restoring the original board. Each bar of length i becomes a rectangle of area $2i$, whichever dimension was doubled. The rectangles still cover each cell exactly once. Place the two rectangles of area $2i$ at turns $2i - 1$ and $2i$, in increasing order of i. The first turn of each pair expands and the second uses its ordinary area. This is a perfect game. □

Example. On a 6×10 board, $K = 10$ and $\gamma = 5$. Halve the height and fill the three rows of width ten with

$$(5, 5), \qquad (4, 3, 2, 1), \qquad (4, 3, 2, 1).$$

Restoring height two to each row gives precisely two rectangles of every even area from 2 through 10.

Formal verification

The archived source [3] proves `PackIt.half_gap_perfect` for arbitrary positive side lengths. Its conclusion requires an ordered legal move list of length K with exactly one covering rectangle at each board cell and zero coverings outside. The proof constructs the repeated area sequence $2, 2, 4, 4, \dots, 2s, 2s$ before deriving legality. The source also proves the triangular characterization of K and Lemma 2. Lean 4.33.1 checks the file with its standard library. Conjecture 6 of [2] proposes a criterion covering all gaps.

References

[1] K. Ando, S. Gervacio and M. Kano, *Disjoint subsets of integers having a constant sum*, Discrete Math. **82** (1990), 7–11. doi:10.1016/0012-365X(90)90040-O.

[2] T. Garrison, M. J. H. Heule and B. Subercaseaux, *PackIt!: Gamified Rectangle Packing*, 12th International Conference on Fun with Algorithms, LIPIcs **291** (2024), 14:1–14:19. doi:10.4230/LIPIcs.FUN.2024.14.

[3] C. Dexter and GPT 6.2 Astra, *Perfect rectangular PackIt! games with half the turns expanded*, Lean source and verification note (2026). Formal proof and verification note.