SHIFT-AND-REVERSE MAXIMIZERS FOR FINITE ODD-CONVEX PERMUTATION SUMS

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ABSTRACT. For any odd function $\varphi\colon \mathbb{Z} \to \mathbb{Q}$ whose successive differences are nondecreasing on the positive integers, we prove that a shift-and-reverse permutation maximizes $\sum_{k=1}^{n} \varphi(k - \sigma(k))$ over S_n. A maximizing shift can always be chosen with $0 \le s \le \lfloor n/3 \rfloor$, and with $s > 0$ when $n \ge 3$. Explicit dual potentials certify optimality on the finite assignment matrix. For every odd power $\varphi(t) = t^q$, the same permutation attains the largest absolute displacement sum. This proves the candidate proposed by Jean-Christophe Pain for general odd powers. The cubic specialization recovers D. C. Blest's extremum and the four-factor formula recorded by V. Jovovic. We also obtain a factored quintic evaluation for each shift. Lean 4 verifies the finite maximum theorem, the odd-power bound, the cubic value at a maximizing shift, and the quintic evaluation.

1. Introduction and Main Results

Displacement statistics of permutations $\sigma \in S_n$—measuring how far elements move from their natural positions—arise throughout combinatorics, probability, and rank statistics. Classical examples include Spearman's footrule $\sum_{k=1}^{n} |k - \sigma(k)|$ studied by Diaconis and Graham [8] (see also Diaconis [7]), the generating function for total displacement [9], and permutation depth [12]. Recently, Pain [11] investigated the extremal problem for the q-th power displacement sum

$$F_{q,n}(\sigma) \coloneqq \sum_{k=1}^{n} (k - \sigma(k))^q, \qquad \sigma \in S_n, \tag{1.1}$$

for positive integers $q, n \ge 1$.

When $q = 2r$ is even, the convexity of $t \mapsto t^{2r}$ on all of $\mathbb{R}$ implies via a classical pairwise exchange (rearrangement) argument that the order-reversing permutation $\sigma_{\mathrm{rev}}(k) = n + 1 - k$ achieves the global maximum of $F_{2r,n}(\sigma)$ over S_n; using Faulhaber's power-sum formulas [10], Pain [11, Theorem 1 and Section 2] obtained closed-form expressions in terms of Bernoulli numbers (for instance, $n(n^2 - 1)/3$ for $q = 2$ and $n(3n^4 - 10n^2 + 7)/15$ for $q = 4$).

When $q = 2r + 1 \ge 3$ is odd, the problem changes qualitatively: $t \mapsto t^q$ is odd, concave on $(-\infty, 0]$, and convex on $[0, \infty)$. In particular, $F_{q,n}(\sigma_{\mathrm{rev}}) = 0$, the pairwise exchange argument breaks down, and maximizing $F_{q,n}(\sigma)$ requires balancing a large number of small negative displacements against a smaller number of large positive displacements. To capture this trade-off, Pain [11, Section 3] introduced the following family of permutations.

Definition 1.1 (Shift-and-reverse permutations)**.** For integers $n \ge 1$ and $0 \le s \le n$, the *shift-and-reverse permutation* $\sigma_{n,s} \in S_n$ is defined by

$$\sigma_{n,s}(k) \coloneqq \begin{cases} k + s & \text{if } 1 \le k \le n - s, \\ n + 1 - k & \text{if } n - s + 1 \le k \le n. \end{cases}$$

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In one-line notation, $\sigma_{n,s} = (1+s, 2+s, \dots, n, s, s-1, \dots, 1)$. Note that $\sigma_{n,0}$ is the identity permutation and $\sigma_{n,n} = \sigma_{\mathrm{rev}}$.

As verified in Lemma 2.3, the displacements $k - \sigma_{n,s}(k)$ consist of $n-s$ copies of $-s$ together with the s values $n-1-2r$ $(0 \le r \le s-1)$, giving

$$F_{q,n}(\sigma_{n,s}) = \sum_{r=0}^{s-1} (n-1-2r)^q - (n-s)\, s^q. \tag{1.2}$$

Pain [11, Sections 3.1–3.3] reports computations for $q = 3, 5$ through $n = 12$ and analyzes the limiting scaling s/n. In Section 3.3 he proposes maximizing over $1 \le s \le \lfloor n/2 \rfloor$ as a candidate for the global maximum magnitude for general odd q. We prove this assertion for every finite n (allowing $s = 0$ in the small cases) and strengthen the shift bound to $s \le \lfloor n/3 \rfloor$.

The finite cubic extremum predates Pain's article. Blest [1] studied sums of cubes of integer differences. His OEIS entry A049031, introduced in 1999, records one third of the cubic maximum and gives the maximizing shift $\lfloor (n+2)/4 \rfloor$. The same entry credits Jovovic (2004) with the four-factor formula using $\lfloor (n+1)/4 \rfloor$ [2]. These shifts differ only when $n \equiv 2 \pmod 4$, when both are optimal. Replacing a permutation by its inverse exchanges the two conventions for the sign of a difference.

There is also a continuous optimal-transport antecedent. For uniform interval marginals and a smooth cost of the form $\psi(x+y)$ with a change from concavity to convexity, Iacò, Tichy and Thonhauser [3, Theorem 3.7] give an optimal map and a dual certificate, drawing on work of Uckelmann [4]. Their map is $\Gamma(x) = \beta - x$ below a threshold β and $\Gamma(x) = x$ above it. Setting $\psi(t) = \varphi(t-1)$ and reflecting the second coordinate gives the shift $x \mapsto x + 1 - \beta$ on the first interval and the reversal $x \mapsto 1 - x$ on the second. The argument below works directly on each finite grid and requires only weak inequalities between successive differences.

Definition 1.2 (Odd-convex functions)**.** A function $\varphi : \mathbb{Z} \to \mathbb{Q}$ is called *odd-convex* if:

(i) $\varphi(-t) = -\varphi(t)$ for all $t \in \mathbb{Z}$; and
(ii) its forward differences $\Delta(t) \coloneqq \varphi(t) - \varphi(t-1)$ are non-decreasing for $t \ge 1$, that is, $\Delta(t) \le \Delta(t+1)$ for all integers $t \ge 1$.

For any $\varphi : \mathbb{Z} \to \mathbb{Q}$ and $n \ge 1$, we write

$$P_{\varphi,n}(\sigma) \coloneqq \sum_{k=1}^{n} \varphi(k - \sigma(k)), \qquad G_{\varphi,n}(s) \coloneqq \sum_{r=0}^{s-1} \varphi(n-1-2r) - (n-s)\, \varphi(s).$$

Our four main theorems are as follows.

Theorem 1.3 (General Odd-Convex Permutation Maximum Theorem)**.** *Let $n \ge 1$ be an integer and let $\varphi : \mathbb{Z} \to \mathbb{Q}$ be any odd-convex function. Then there exists an integer s with $0 \le s \le \lfloor n/3 \rfloor$ (and $1 \le s \le \lfloor n/3 \rfloor$ whenever $n \ge 3$) such that*

$$\max_{\sigma \in S_n} P_{\varphi,n}(\sigma) = P_{\varphi,n}(\sigma_{n,s}) = G_{\varphi,n}(s) = \sum_{r=0}^{s-1} \varphi(n-1-2r) - (n-s)\, \varphi(s).$$

Specializing Theorem 1.3 to $\varphi(t) = t^q$ and invoking the inversion symmetry $F_{q,n}(\sigma^{-1}) = -F_{q,n}(\sigma)$ gives the bound for every odd power.

Theorem 1.4 (Maximum absolute odd-power displacement)**.** *For every odd integer $q \ge 1$ and every integer $n \ge 1$, there exists an integer s with $0 \le s \le \lfloor n/3 \rfloor$ such that for all $\sigma \in S_n$,*

$$\left| F_{q,n}(\sigma) \right| \le F_{q,n}(\sigma_{n,s}) = \sum_{r=0}^{s-1} (n-1-2r)^q - (n-s)\, s^q.$$

For $q = 3$ the finite theorem recovers Blest's extremum [1]. The next calculation identifies all maximizing shifts and gives the four-factor expression recorded in A049031 [2].

Theorem 1.5 (Exact Closed Form for $q = 3$)**.** *For every integer $n \geq 1$ and every $0 \leq s \leq n$,*

$$F_{3,n}(\sigma_{n,s}) = s(n-s-1)(n-s)(n-s+1) = 6s\binom{n-s+1}{3}. \tag{1.3}$$

Setting $s^* := \lfloor (n+1)/4 \rfloor$, for every $n \geq 1$ we have

$$\max_{\sigma \in S_n} \left| \sum_{k=1}^{n} (k - \sigma(k))^3 \right| = F_{3,n}(\sigma_{n,s^*}) = s^*(n-s^*-1)(n-s^*)(n-s^*+1).$$

For $n = 1, 2$, every $s \in \{0, \ldots, n\}$ attains the value zero. For $n \geq 3$, s^* is the unique maximizing shift in this set unless $n \equiv 2 \pmod 4$. In that case the two maximizing shifts are $(n-2)/4$ and $(n+2)/4$.

For $q = 5$, we obtain an analogous exact factored polynomial formula.

Theorem 1.6 (Exact Factored Formula for $q = 5$)**.** *For every integer $n \geq 1$ and every $0 \leq s \leq n$,*

$$3\,F_{5,n}(\sigma_{n,s}) = s(n-s)\big((n-s)^2 - 1\big)\big(3(n-s)^2 + 10s^2 - 7\big). \tag{1.4}$$

Proof architecture and organization. The central obstacle in Pain's problem is that local pairwise swaps in S_n can either increase or decrease $P_{\varphi,n}(\sigma)$ depending on the signs of the displacements, preventing a greedy exchange proof. Instead, we view $\max_{\sigma \in S_n} \sum_{k=1}^{n} \varphi(k - \sigma(k))$ as a maximum-weight perfect matching (linear assignment) problem on the complete bipartite graph $K_{n,n}$ with edge weights $W_{k,j} = \varphi(k-j)$. By linear programming weak duality, optimality of $\sigma_{n,s}$ follows immediately once we exhibit explicit dual vertex potentials $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{Q}$ satisfying

$$u_k + v_j \geq \varphi(k-j) \quad \forall k, j \in \{1, \ldots, n\}, \qquad \text{and} \qquad \sum_{k=1}^{n} u_k + \sum_{j=1}^{n} v_j = G_{\varphi,n}(s).$$

In Section 2 we establish the discrete convexity properties of odd-convex functions and prove the unconditional range reduction $1 \leq s \leq \lfloor n/3 \rfloor$ for $n \geq 3$. In Section 3 we construct the dual parameters (c, A, B) and potentials (u_k, v_j) explicitly in closed form, verify complementary slackness, establish four-quadrant feasibility, and complete the proofs of Theorems 1.3 and 1.4. Section 4 proves Theorems 1.5 and 1.6, tabulates exact maxima for $1 \leq n \leq 12$, and compares the quintic values with [11, p. 8]. Finally, Section 5 lists the statements checked by Lean 4.33.1.

2. Odd-Convex Functions and Unconditional Range Reduction

Throughout this section, let $\varphi : \mathbb{Z} \to \mathbb{Q}$ be an odd-convex function with forward differences $\Delta(t) = \varphi(t) - \varphi(t-1)$.

Lemma 2.1 (Basic properties and discrete secant inequality)**.** *Let $\varphi : \mathbb{Z} \to \mathbb{Q}$ be odd-convex.*

(i) $\varphi(0) = 0$.
(ii) *For all integers $0 \leq a \leq b$, $\varphi(b) - \varphi(a) = \sum_{t=a+1}^{b} \Delta(t)$; in particular, $\varphi(m) = \sum_{t=1}^{m} \Delta(t)$ for all $m \geq 0$.*
(iii) *For all integers $1 \leq a \leq b$, $\Delta(a) \leq \Delta(b)$.*
(iv) (Discrete secant inequality.) *For all integers $0 \leq a \leq b$,*

$$b\,\varphi(a) \leq a\,\varphi(b). \tag{2.1}$$

Proof. Part (i) follows immediately from $\varphi(-0) = -\varphi(0)$. Part (ii) is a telescoping sum using $\varphi(0) = 0$, and part (iii) follows from Definition 1.2(ii) by induction on $b - a$.

For (iv), if $a = 0$ then both sides vanish by (i). Suppose $1 \le a \le b$. Using (ii), we write

$$b\,\varphi(a) - a\,\varphi(b) = (b-a)\,\varphi(a) - a\big(\varphi(b) - \varphi(a)\big) = (b-a)\sum_{t=1}^{a}\Delta(t) - a\sum_{t=a+1}^{b}\Delta(t).$$

By (iii), every summand in the first sum satisfies $\Delta(t) \le \Delta(a)$, whereas every summand in the second sum satisfies $\Delta(t) \ge \Delta(a)$. Therefore,

$$(b-a)\sum_{t=1}^{a}\Delta(t) \le (b-a)\,a\,\Delta(a) \le a\sum_{t=a+1}^{b}\Delta(t),$$

which yields $b\,\varphi(a) \le a\,\varphi(b)$. □

Lemma 2.2 (Odd powers are odd-convex)**.** *For every odd integer $q \ge 1$, the map $\varphi_q(t) \coloneqq t^q$ is odd-convex.*

Proof. Since q is odd, $(-t)^q = -t^q$ for all $t \in \mathbb{Z}$. For $t \ge 1$, the monotonicity condition $\Delta(t) \le \Delta(t+1)$ is equivalent to

$$(t+1)^q + (t-1)^q - 2t^q \ge 0.$$

We prove by induction on $m \ge 1$ that $(t+1)^m + (t-1)^m \ge 2t^m$ for all integers $t \ge 1$ (indeed for every integer $m \ge 1$). For $m = 1$, $(t+1) + (t-1) = 2t$. Assuming the inequality for $m \ge 1$, we use the algebraic identity

$$(t+1)^{m+1} + (t-1)^{m+1} - 2t^{m+1} = t\big((t+1)^m + (t-1)^m - 2t^m\big) + \big((t+1)^m - (t-1)^m\big).$$

Since $t \ge 1$, both terms on the right-hand side are non-negative (as $(t+1)^m \ge (t-1)^m \ge 0$ for $t \ge 1$), completing the induction. □

We now evaluate $P_{\varphi,n}$ on the shift-and-reverse permutations $\sigma_{n,s}$ and compute its discrete derivative in s.

Lemma 2.3 (Evaluation and step difference of $G_{\varphi,n}$)**.** *Let $\varphi : \mathbb{Z} \to \mathbb{Q}$ be odd-convex and let $n \ge 1$.*

(i) *For every integer $0 \le s \le n$,*

$$P_{\varphi,n}(\sigma_{n,s}) = G_{\varphi,n}(s) \coloneqq \sum_{r=0}^{s-1}\varphi(n-1-2r) - (n-s)\,\varphi(s).$$

(ii) *For every integer $0 \le s \le n-1$,*

$$G_{\varphi,n}(s+1) - G_{\varphi,n}(s) = \varphi(n-2s-1) + \varphi(s) - (n-s-1)\,\Delta(s+1). \tag{2.2}$$

Proof. For (i), split the sum over $k \in \{1, \dots, n\}$ into $1 \le k \le n-s$ and $n-s+1 \le k \le n$. On the first block, $k - \sigma_{n,s}(k) = -s$, contributing $(n-s)\varphi(-s) = -(n-s)\varphi(s)$ by oddness. On the second block, writing $k = n - r$ with $0 \le r \le s-1$, we have $\sigma_{n,s}(n-r) = r+1$, so $k - \sigma_{n,s}(k) = n-1-2r$. Summing both blocks gives $G_{\varphi,n}(s)$.

For (ii), subtracting $G_{\varphi,n}(s)$ from $G_{\varphi,n}(s+1)$ yields

$$\begin{aligned} G_{\varphi,n}(s+1) - G_{\varphi,n}(s) &= \varphi(n-1-2s) - (n-s-1)\,\varphi(s+1) + (n-s)\,\varphi(s) \\ &= \varphi(n-2s-1) + \varphi(s) - (n-s-1)\big(\varphi(s+1) - \varphi(s)\big), \end{aligned}$$

which is (2.2). □

Pain's search uses $1 \le s \le \lfloor n/2 \rfloor$. The smallest displacement in its reversed block is $n - 2s + 1$, which is positive precisely when $2s \le n$ and nonnegative precisely when $2s \le n+1$. Within Pain's range, the following comparison reduces the search further.

Proposition 2.4 (Unconditional range reduction to $1 \le s \le \lfloor n/3 \rfloor$)**.** *Let $\varphi : \mathbb{Z} \to \mathbb{Q}$ be odd-convex and let $n \ge 3$.*

(i) $G_{\varphi,n}(0) = 0 \le G_{\varphi,n}(1)$.
(ii) *If $s \ge 1$ satisfies $2s \le n < 3s$, then $G_{\varphi,n}(s) \le G_{\varphi,n}(s-1)$.*
(iii) *There exists an integer s with $1 \le s \le \lfloor n/3 \rfloor$ such that*

$$G_{\varphi,n}(s') \le G_{\varphi,n}(s) \qquad \text{for all } 0 \le s' \le \lfloor n/2 \rfloor.$$

Proof. For (i), $G_{\varphi,n}(0) = -n\varphi(0) = 0$. Since $n \ge 3$, Lemma 2.1(ii)–(iii) gives

$$G_{\varphi,n}(1) = \varphi(n-1) - (n-1)\,\varphi(1) = \sum_{t=1}^{n-1} (\Delta(t) - \Delta(1)) \ge 0.$$

For (ii), let $s \ge 1$ satisfy $2s \le n < 3s$, and set $a \coloneqq n-2s+1$. Then $1 \le a \le s$ and $a+(s-1) = n-s$. Applying (2.2) at $s-1$ together with Lemma 2.1(ii)–(iii) yields

$$\begin{aligned} G_{\varphi,n}(s) - G_{\varphi,n}(s-1) &= \varphi(a) + \varphi(s-1) - (n-s)\,\Delta(s) \\ &= \sum_{t=1}^{a} (\Delta(t) - \Delta(s)) + \sum_{t=1}^{s-1} (\Delta(t) - \Delta(s)) \le 0, \end{aligned}$$

since every index t in both sums satisfies $1 \le t \le s$, so $\Delta(t) \le \Delta(s)$.

For (iii), choose a maximizer $s_0 \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$ of $G_{\varphi,n}$. If $s_0 = 0$, we may replace s_0 by $s = 1 \le \lfloor n/3 \rfloor$ by (i). If $s_0 > \lfloor n/3 \rfloor$, repeated application of (ii) gives $G_{\varphi,n}(s_0) \le G_{\varphi,n}(s_0 - 1) \le \cdots \le G_{\varphi,n}(\lfloor n/3 \rfloor)$, so $s = \lfloor n/3 \rfloor \ge 1$ is also a maximizer over $\{0, \dots, \lfloor n/2 \rfloor\}$. □

3. Explicit Linear Programming Duality Certificate

Throughout this section, fix an odd-convex function $\varphi : \mathbb{Z} \to \mathbb{Q}$, an integer $n \ge 3$, and a maximizer $s \in \{1, \dots, \lfloor n/3 \rfloor\}$ of $G_{\varphi,n}$ over $\{0, \dots, \lfloor n/2 \rfloor\}$ as guaranteed by Proposition 2.4(iii). For brevity we write $G(r) \coloneqq G_{\varphi,n}(r)$, and define the auxiliary quantities

$$S \coloneqq \varphi(n-2s+1)+\varphi(s), \quad A_{\max} \coloneqq \Delta(n-2s+1), \quad B_{\max} \coloneqq \Delta(n-2s), \quad D \coloneqq A_{\max}+B_{\max}, \tag{3.1}$$

together with $m \coloneqq n - s - 1 \ge 1$ (since $n \ge 3s \ge 3$, we have $n - s - 1 \ge 2s - 1 \ge 1$). Notice that $n - 2s \ge s \ge 1$, so $A_{\max}$ and $B_{\max}$ are well-defined with $B_{\max} \le A_{\max}$, and

$$S - D = \varphi(n - 2s + 1) - \Delta(n - 2s + 1) - \Delta(n - 2s) + \varphi(s) = \varphi(n - 2s - 1) + \varphi(s). \tag{3.2}$$

Definition 3.1 (Dual slope and transition parameters)**.** Define $(c, A, B) \in \mathbb{Q}^3$ in two cases:

- **Case 1** ($S - D \ge m\,\Delta(s)$):

$$c \coloneqq \frac{S - D}{m}, \qquad A \coloneqq A_{\max}, \qquad B \coloneqq B_{\max}.$$

- **Case 2** ($S - D < m\,\Delta(s)$):

$$c \coloneqq \Delta(s), \qquad A \coloneqq \min\big(A_{\max},\, S - (n-s)\,\Delta(s)\big), \qquad B \coloneqq S - m\,\Delta(s) - A.$$

Proposition 3.2 (Dual parameter invariants)**.** *With $n \ge 3$ and $s \in \{1, \dots, \lfloor n/3 \rfloor\}$ maximizing G on $\{0, \dots, \lfloor n/2 \rfloor\}$, the parameters (c, A, B) of Definition 3.1 satisfy all seven relations:*

$$A + B + (n - s - 1)\,c = S = \varphi(n - 2s + 1) + \varphi(s), \tag{3.3}$$

$$\Delta(s) \le c \le \Delta(s + 1), \tag{3.4}$$

$$c \le A \le A_{\max} = \Delta(n - 2s + 1), \qquad c \le B \le B_{\max} = \Delta(n - 2s). \tag{3.5}$$

Proof. We first derive the left- and right-optimality bounds on S. Since $0 \le s-1 \le \lfloor n/2 \rfloor$, optimality of s gives $G(s) - G(s-1) \ge 0$. By (2.2) at $s-1$ and $\varphi(s-1) = \varphi(s) - \Delta(s)$,

$$0 \le G(s) - G(s-1) = \varphi(n-2s+1) + \varphi(s-1) - (n-s)\,\Delta(s) = S - (n-s+1)\,\Delta(s),$$

so

$$S \ge (n-s+1)\,\Delta(s) = (m+2)\,\Delta(s). \tag{3.6}$$

Next, we claim the right-optimality bound

$$S - D \le m\,\Delta(s+1). \tag{3.7}$$

Indeed, if $n = 3$, then $3s \le 3$ forces $s = 1$ and $m = 1$, so by (3.2),

$$S - D = \varphi(0) + \varphi(1) = \Delta(1) \le \Delta(2) = m\,\Delta(s+1).$$

If $n \ge 4$, then $3s \le n$ implies $2(s+1) \le n$, i.e., $s+1 \le \lfloor n/2 \rfloor$; optimality of s yields $G(s+1) - G(s) \le 0$, which by (2.2) and (3.2) is precisely (3.7).

We also observe that whenever $n = 3s$, we have $n - 2s - 1 = s - 1$ and $m = 2s - 1 = (s-1) + s$, so Lemma 2.1(ii)–(iii) gives

$$S - D = \varphi(s-1) + \varphi(s) = \sum_{t=1}^{s-1} \Delta(t) + \sum_{t=1}^{s} \Delta(t) \le (s-1)\,\Delta(s) + s\,\Delta(s) = m\,\Delta(s). \tag{3.8}$$

Finally, since $n - 2s \ge s \ge 1$, Lemma 2.1(iii) yields $\Delta(s) \le B_{\max} \le A_{\max}$.

Verification in Case 1 ($S - D \ge m\,\Delta(s)$). Here $A = A_{\max}$, $B = B_{\max}$, and $c = (S-D)/m$. Then $A + B + mc = D + (S - D) = S$, giving (3.3). Dividing $m\,\Delta(s) \le S - D \le m\,\Delta(s+1)$ (from the Case 1 hypothesis and (3.7)) by $m > 0$ gives (3.4). For (3.5), we need only check $c \le B_{\max}$ (since $B_{\max} \le A_{\max}$). If $n = 3s$, then (3.8) combined with $S - D \ge m\,\Delta(s)$ forces $c = \Delta(s) = \Delta(n-2s) = B_{\max}$. If $n > 3s$, then $n - 2s \ge s+1$, so $c \le \Delta(s+1) \le \Delta(n-2s) = B_{\max}$.

Verification in Case 2 ($S - D < m\,\Delta(s)$). Here $c = \Delta(s)$, $A = \min(A_{\max}, S - (n-s)\Delta(s))$, and $B = S - m\Delta(s) - A$. Equation (3.3) holds identically by definition of B, and (3.4) holds because $\Delta(s) \le \Delta(s+1)$. By (3.6), $S - (n-s)\Delta(s) \ge \Delta(s)$, and we already have $A_{\max} \ge B_{\max} \ge \Delta(s)$; hence $c = \Delta(s) \le A \le A_{\max}$. For B, since $A \le S - (n-s)\Delta(s) = S - (m+1)\Delta(s)$, we obtain $B = S - m\Delta(s) - A \ge \Delta(s) = c$. Lastly, to verify $B \le B_{\max}$:

- if $A = A_{\max}$, then $B = S - m\Delta(s) - A_{\max} < D - A_{\max} = B_{\max}$ by the Case 2 condition $S - D < m\Delta(s)$;
- if $A = S - (n-s)\Delta(s)$, then $B = (n - s - m)\Delta(s) = \Delta(s) \le B_{\max}$.

This completes the proof of all seven invariants. □

We now construct the explicit dual potentials (u_k, v_j) on $K_{n,n}$.

Definition 3.3 (Explicit dual potentials)**.** Using (c, A, B) from Definition 3.1, define $u_1, \dots, u_n \in \mathbb{Q}$ and $v_1, \dots, v_n \in \mathbb{Q}$ as follows:

- For $n - s + 1 \le k \le n$ (writing $k = n - a$ with $0 \le a \le s-1$):

$$u_{n-a} \coloneqq \varphi(n-1) - \sum_{i=0}^{a-1} \Delta(n-1-2i).$$

- For $1 \le k \le n - s$:

$$u_k \coloneqq u_{n-s+1} - A - (n-s-k)\,c.$$

- For $1 \le j \le s$ (writing $j = b + 1$ with $0 \le b \le s-1$):

$$v_{b+1} \coloneqq -\sum_{i=0}^{b-1} \Delta(n-2-2i).$$

- For $s+1 \le j \le n$:
$$v_j \coloneqq v_s - B - (j-s-1)\,c.$$

(In the Lean 4 formalization of Section 5, the same potentials are indexed by $k_0 = k-1, j_0 = j-1 \in \operatorname{Fin} n$, which leaves all differences $k-j = k_0 - j_0$ invariant.)

The index set $\{1,\dots,n\}^2$ naturally partitions into four quadrants along the thresholds $k \le n-s$ and $j \le s$:

$$Q_1 \coloneqq \{1,\dots,n-s\} \times \{s+1,\dots,n\}, \qquad Q_2 \coloneqq \{1,\dots,n-s\} \times \{1,\dots,s\},$$
$$Q_3 \coloneqq \{n-s+1,\dots,n\} \times \{s+1,\dots,n\}, \qquad Q_4 \coloneqq \{n-s+1,\dots,n\} \times \{1,\dots,s\}.$$

Notice that the graph of $\sigma_{n,s}$ lies entirely in $Q_1 \cup Q_4$: the shift pairs $(k, k+s)$ $(1 \le k \le n-s)$ lie in Q_1, and the reversal pairs $(n-r, r+1)$ $(0 \le r \le s-1)$ lie in Q_4. We next verify complementary slackness on both pieces and identify $u_k + v_j$ explicitly on Q_1.

Lemma 3.4 (Telescoping, Quadrant 1 formula, and complementary slackness)**.** *Define the affine function $L(d) \coloneqq -\varphi(s) + (d+s)\,c$ for $d \in \mathbb{Z}$. Then:*

(i) *For every integer $r \ge 0$ with $2r \le n-1$,*
$$\sum_{i=0}^{r-1} \big(\Delta(n-1-2i) + \Delta(n-2-2i)\big) = \varphi(n-1) - \varphi(n-1-2r). \tag{3.9}$$
In particular, for every $0 \le r \le s-1$, $u_{n-r} + v_{r+1} = \varphi(n-1-2r)$.

(ii) *For every $(k,j) \in Q_1$, $u_k + v_j = L(k-j)$. In particular, for every $1 \le k \le n-s$, $u_k + v_{k+s} = -\varphi(s) = \varphi(-s)$.*

(iii) *The total dual objective matches $G(s)$:*
$$\sum_{k=1}^{n} u_k + \sum_{j=1}^{n} v_j = G(s).$$

Proof. For (i), each summand satisfies

$$\Delta(n-1-2i) + \Delta(n-2-2i) = \varphi(n-1-2i) - \varphi(n-3-2i),$$

so (3.9) telescopes by induction on r. For $0 \le r \le s-1$, we have $2r \le 2s-2 \le n-1$, and Definition 3.3 gives

$$u_{n-r} + v_{r+1} = \varphi(n-1) - \sum_{i=0}^{r-1} \big(\Delta(n-1-2i) + \Delta(n-2-2i)\big) = \varphi(n-1-2r).$$

For (ii), let $(k,j) \in Q_1$, so $1 \le k \le n-s$ and $s+1 \le j \le n$. Applying part (i) at $r = s-1$ gives $u_{n-s+1} + v_s = \varphi(n-2s+1)$. By Definition 3.3 and (3.3),

$$\begin{aligned} u_k + v_j &= (u_{n-s+1} + v_s) - A - B - (n-s-k+j-s-1)\,c \\ &= \varphi(n-2s+1) - \big(\varphi(n-2s+1) + \varphi(s) - (n-s-1)\,c\big) - (n-2s-1-k+j)\,c \\ &= -\varphi(s) + (k-j+s)\,c = L(k-j). \end{aligned}$$

Setting $j = k+s$ gives $u_k + v_{k+s} = L(-s) = -\varphi(s) = \varphi(-s)$.

For (iii), since $\sigma_{n,s}$ is a bijection on $\{1,\dots,n\}$, parts (i) and (ii) together with Lemma 2.3(i) give

$$\sum_{k=1}^{n} u_k + \sum_{j=1}^{n} v_j = \sum_{k=1}^{n} \big(u_k + v_{\sigma_{n,s}(k)}\big) = \sum_{k=1}^{n-s} \varphi(-s) + \sum_{r=0}^{s-1} \varphi(n-1-2r) = G(s). \qquad \square$$

We now prove dual feasibility across all four quadrants in three clean lemmas.

Lemma 3.5 (Linear majorant on $d \le n-2s-1$)**.** *Set $d_0 := n-2s-1 \ge 0$. For every integer $d \le d_0$,*

$$\varphi(d) \le L(d) = -\varphi(s) + (d+s)\,c.$$

Proof. We consider three ranges for d.

Subcase 1: $d \le -s$. Write $d = -(s+m')$ with $m' \ge 0$. By oddness of φ, Lemma 2.1(ii)–(iii), and $c \le \Delta(s+1)$ from (3.4),

$$L(d) - \varphi(d) = \varphi(s+m') - \varphi(s) - m'c = \sum_{t=s+1}^{s+m'} (\Delta(t) - c) \ge 0.$$

Subcase 2: $-s < d \le 0$. Write $d = -a$ with $0 \le a < s$. By oddness of φ, Lemma 2.1(ii)–(iii), and $\Delta(s) \le c$ from (3.4),

$$L(d) - \varphi(d) = (s-a)\,c - (\varphi(s) - \varphi(a)) = \sum_{t=a+1}^{s} (c - \Delta(t)) \ge 0.$$

Subcase 3: $0 < d \le d_0$. In this subcase necessarily $d_0 \ge 1$. At the left endpoint 0, Subcase 2 gives

$$L(0) = -\varphi(s) + s\,c = \sum_{t=1}^{s} (c - \Delta(t)) \ge 0.$$

At the right endpoint d_0, since $d_0 + s = n - s - 1$, (3.3), (3.5), and (3.2) give

$$L(d_0) = -\varphi(s) + (n-s-1)\,c = \varphi(n-2s+1) - A - B \ge \varphi(n-2s+1) - A_{\max} - B_{\max} = \varphi(d_0).$$

Combining $L(0) \ge 0$, $L(d_0) \ge \varphi(d_0)$, and the discrete secant inequality $d_0\,\varphi(d) \le d\,\varphi(d_0)$ from Lemma 2.1(iv), we obtain

$$d_0\,\varphi(d) \le d\,\varphi(d_0) + (d_0 - d)\cdot 0 \le d\,L(d_0) + (d_0 - d)\,L(0) = d_0\,L(d),$$

where the last equality holds because L is affine. Dividing by $d_0 > 0$ yields $\varphi(d) \le L(d)$. □

Lemma 3.6 (Quadrant 4 feasibility)**.** *For every $(k,j) \in Q_4$ (i.e., $n-s+1 \le k \le n$ and $1 \le j \le s$),*

$$\varphi(k-j) \le u_k + v_j.$$

Proof. Write $k = n-a$ and $j = b+1$ with $0 \le a, b \le s-1$, so $k - j = n-1-a-b \ge n+1-2s > 0$.

If $a \le b$, we expand from the matched pair $(n-a, a+1)$ using Lemma 3.4(i):

$$u_{n-a} + v_{b+1} = (u_{n-a} + v_{a+1}) - \sum_{i=a}^{b-1} \Delta(n-2-2i) = \varphi(n-1-2a) - \sum_{i=a}^{b-1} \Delta(n-2-2i).$$

On the other hand, by Lemma 2.1(ii) and the index substitution $t = n-1-a-i$,

$$\varphi(n-1-2a) - \varphi(n-1-a-b) = \sum_{t=n-a-b}^{n-1-2a} \Delta(t) = \sum_{i=a}^{b-1} \Delta(n-1-a-i).$$

For every $a \le i \le b-1$, we have $n-2-2i \ge n-2s \ge s \ge 1$ and $n-2-2i < n-1-a-i$ (since $i \ge a$), whence $\Delta(n-2-2i) \le \Delta(n-1-a-i)$ by Lemma 2.1(iii). Summing over $i \in \{a, \dots, b-1\}$ yields $\varphi(n-1-a-b) \le u_{n-a} + v_{b+1}$.

Symmetrically, if $b \le a$, we expand from the matched pair $(n-b, b+1)$:

$$u_{n-a} + v_{b+1} = (u_{n-b} + v_{b+1}) - \sum_{i=b}^{a-1} \Delta(n-1-2i) = \varphi(n-1-2b) - \sum_{i=b}^{a-1} \Delta(n-1-2i),$$

while

$$\varphi(n-1-2b) - \varphi(n-1-a-b) = \sum_{i=b}^{a-1} \Delta(n-1-b-i).$$

For every $b \le i \le a-1$, we have $1 \le n-1-2i \le n-1-b-i$, so $\Delta(n-1-2i) \le \Delta(n-1-b-i)$, and summing again gives $\varphi(n-1-a-b) \le u_{n-a} + v_{b+1}$. □

Proposition 3.7 (Full four-quadrant dual feasibility)**.** *For all $k, j \in \{1, \dots, n\}$,*

$$\varphi(k-j) \le u_k + v_j.$$

Proof. We check each of the four quadrants Q_1, Q_2, Q_3, Q_4.

Quadrant 1 ($(k,j) \in Q_1$): Here $k \le n-s$ and $j \ge s+1$, so $k-j \le n-2s-1 = d_0$. Lemma 3.4(ii) and Lemma 3.5 give $u_k + v_j = L(k-j) \ge \varphi(k-j)$.

Quadrant 4 ($(k,j) \in Q_4$): This is precisely Lemma 3.6.

Quadrant 2 ($1 \le k \le n-s$ and $1 \le j \le s$): We split according to whether $k-j \le d_0$ or $k-j \ge n-2s$.

- *Low difference ($k-j \le n-2s-1$):* Compare v_j with v_{s+1}. By Definition 3.3,

$$v_j - v_{s+1} = B + \sum_{i=j-1}^{s-2} \Delta(n-2-2i).$$

There are $s-j$ terms in the sum, and for every $i \le s-2$ we have $n-2-2i \ge n-2s+2 \ge s+2 > s+1$, so $\Delta(n-2-2i) \ge \Delta(s+1) \ge c$ by (3.4). Since also $B \ge c$ by (3.5), we obtain $v_j - v_{s+1} \ge (s+1-j)\,c$. Because $(k, s+1) \in Q_1$, Lemma 3.4(ii) and Lemma 3.5 yield

$$u_k + v_j \ge (u_k + v_{s+1}) + (s+1-j)\,c = L(k-s-1) + (s+1-j)\,c = L(k-j) \ge \varphi(k-j).$$

- *High difference ($k-j \ge n-2s$):* Step down in k from the Quadrant 4 pair $(n-s+1, j) \in Q_4$. By Definition 3.3 and Lemma 3.6,

$$u_k + v_j = (u_{n-s+1} + v_j) - A - (n-s-k)\,c \ge \varphi(n-s+1-j) - A - (n-s-k)\,c.$$

By Lemma 2.1(ii),

$$\varphi(n-s+1-j) - \varphi(k-j) = \Delta(n-s+1-j) + \sum_{t=k-j+1}^{n-s-j} \Delta(t).$$

Since $j \le s$, the top term satisfies $n-s+1-j \ge n-2s+1$, so $\Delta(n-s+1-j) \ge A_{\max} \ge A$ by (3.5). Each of the $n-s-k$ remaining terms has index $t \ge k-j+1 \ge n-2s+1 \ge s+1$ (as $3s \le n$), so $\Delta(t) \ge \Delta(s+1) \ge c$ by (3.4). Hence $\varphi(n-s+1-j) - \varphi(k-j) \ge A + (n-s-k)\,c$, giving $u_k + v_j \ge \varphi(k-j)$.

Quadrant 3 ($n-s+1 \le k \le n$ and $s+1 \le j \le n$): Again we split at $d_0 = n-2s-1$.

- *Low difference ($k-j \le n-2s-1$):* Compare u_k with u_{n-s}. By Definition 3.3,

$$u_k - u_{n-s} = A + \sum_{i=n-k}^{s-2} \Delta(n-1-2i).$$

Each of the $k-(n-s+1)$ summands has $n-1-2i \ge n-2s+3 \ge s+3 > s+1$, so $\Delta(n-1-2i) \ge \Delta(s+1) \ge c$, and $A \ge c$ by (3.5). Thus $u_k - u_{n-s} \ge (k-n+s)\,c$. Since $(n-s, j) \in Q_1$, Lemma 3.4(ii) and Lemma 3.5 give

$$u_k + v_j \ge (u_{n-s} + v_j) + (k-n+s)\,c = L(n-s-j) + (k-n+s)\,c = L(k-j) \ge \varphi(k-j).$$

- *High difference ($k-j \ge n-2s$):* Step down in j from the Quadrant 4 pair $(k, s) \in Q_4$. By Definition 3.3 and Lemma 3.6,

$$u_k + v_j = (u_k + v_s) - B - (j-s-1)\,c \ge \varphi(k-s) - B - (j-s-1)\,c.$$

By Lemma 2.1(ii),

$$\varphi(k-s) - \varphi(k-j) = \Delta(k-s) + \sum_{t=k-j+1}^{k-s-1} \Delta(t).$$

Since $k \geq n-s+1$, we have $k-s \geq n-2s+1 > n-2s$, so $\Delta(k-s) \geq B_{\max} \geq B$ by (3.5). Each of the $j-s-1$ remaining summands has $t \geq k-j+1 \geq n-2s+1 \geq s+1$, so $\Delta(t) \geq \Delta(s+1) \geq c$. Consequently $\varphi(k-s) - \varphi(k-j) \geq B + (j-s-1)\,c$, and therefore $u_k + v_j \geq \varphi(k-j)$. □

We can now prove Theorems 1.3 and 1.4 in just a few lines.

Proof of Theorem 1.3. If $n \in \{1,2\}$, take $s = 0 = \lfloor n/3 \rfloor$. Then $P_{\varphi,n}(\sigma_{n,0}) = G_{\varphi,n}(0) = 0$. For $n = 1$, the unique permutation satisfies $P_{\varphi,1}(\mathrm{id}) = \varphi(0) = 0$; for $n = 2$, the identity gives $2\varphi(0) = 0$ and the transposition gives $\varphi(-1) + \varphi(1) = 0$ by oddness.

If $n \geq 3$, Proposition 2.4(iii) furnishes $s \in \{1, \dots, \lfloor n/3 \rfloor\}$ maximizing $G_{\varphi,n}$ on $\{0, \dots, \lfloor n/2 \rfloor\}$. Let (u_k, v_j) be the dual potentials of Definition 3.3. For every permutation $\sigma \in S_n$, Proposition 3.7, re-indexing $j = \sigma(k)$, Lemma 3.4(iii), and Lemma 2.3(i) give

$$P_{\varphi,n}(\sigma) = \sum_{k=1}^{n} \varphi(k - \sigma(k)) \leq \sum_{k=1}^{n} (u_k + v_{\sigma(k)}) = \sum_{k=1}^{n} u_k + \sum_{j=1}^{n} v_j = G_{\varphi,n}(s) = P_{\varphi,n}(\sigma_{n,s}). \qquad \square$$

Proof of Theorem 1.4. By Lemma 2.2, $\varphi_q(t) = t^q$ is odd-convex for every odd integer $q \geq 1$. Applying Theorem 1.3 to φ_q gives $0 \leq s \leq \lfloor n/3 \rfloor$ such that $F_{q,n}(\tau) \leq F_{q,n}(\sigma_{n,s})$ for all $\tau \in S_n$. Moreover, for any $\sigma \in S_n$, setting $j = \sigma(k)$ gives

$$F_{q,n}(\sigma^{-1}) = \sum_{j=1}^{n} (j - \sigma^{-1}(j))^q = \sum_{k=1}^{n} (\sigma(k) - k)^q = -\sum_{k=1}^{n} (k - \sigma(k))^q = -F_{q,n}(\sigma),$$

since q is odd. Applying the upper bound to both $\tau = \sigma$ and $\tau = \sigma^{-1}$ yields

$$-F_{q,n}(\sigma_{n,s}) \leq F_{q,n}(\sigma) \leq F_{q,n}(\sigma_{n,s}),$$

that is, $|F_{q,n}(\sigma)| \leq F_{q,n}(\sigma_{n,s})$. □

4. Exact Closed Forms for $q = 3$ and $q = 5$

We now specialize (1.2) to $q = 3$ and $q = 5$ to prove Theorems 1.5 and 1.6.

Proof of Theorem 1.5. First, we claim that for all integers n and $k \geq 0$,

$$\sum_{r=0}^{k-1} (n - 1 - 2r)^3 = k(n-k)\big((n-k)^2 + k^2 - 1\big). \tag{4.1}$$

For $k = 0$ both sides vanish. Assuming (4.1) for k, a direct polynomial expansion verifies that

$$(k+1)(n-k-1)\big((n-k-1)^2 + (k+1)^2 - 1\big) - k(n-k)\big((n-k)^2 + k^2 - 1\big) = (n-1-2k)^3,$$

establishing (4.1) by induction on k. Setting $k = s$ and subtracting $(n-s)s^3$ yields

$$\begin{aligned} F_{3,n}(\sigma_{n,s}) &= s(n-s)\big((n-s)^2 + s^2 - 1\big) - (n-s)s^3 \\ &= s(n-s)\big((n-s)^2 - 1\big) = s(n-s-1)(n-s)(n-s+1) = 6s\binom{n-s+1}{3}, \end{aligned}$$

which proves (1.3).

To maximize $g(a) \coloneqq a(n-a-1)(n-a)(n-a+1)$ over $0 \leq a \leq n$, compute its forward difference:

$$g(a+1) - g(a) = (n-a-1)(n-a)(n-4a-2). \tag{4.2}$$

Set $s^* \coloneqq \lfloor (n+1)/4 \rfloor \leq n$. For every integer a with $0 \leq a < s^*$, we have $a + 1 \leq (n+1)/4$, whence $4a + 2 \leq n - 1 < n$, so $n - 4a - 2 > 0$ and $n - a - 1 \geq 0$; thus $g(a) \leq g(a+1)$ (strictly when $a \leq n-2$). Conversely, for every integer a with $s^* \leq a \leq n-1$, we have $4a + 4 > n + 1$, i.e., $n - 4a - 2 \leq 0$, so $g(a+1) \leq g(a)$ by (4.2). Hence $g(a) \leq g(s^*)$ for all $0 \leq a \leq n$, and Theorem 1.4 immediately implies

$$\max_{\sigma \in S_n} |F_{3,n}(\sigma)| = F_{3,n}(\sigma_{n,s^*}) = g(s^*).$$

For $n = 1, 2$ the product defining g vanishes at every admissible shift. For $n \geq 3$, $g(n-1) = g(n) = 0 < g(1)$. On the remaining steps $0 \leq a \leq n-2$ the first two factors in (4.2) are positive. The difference therefore vanishes there exactly when $n = 4a+2$, and it changes from positive to negative across the peak. This proves the claimed uniqueness and identifies both shifts in the tie case. □

Proof of Theorem 1.6. We first establish by induction on $k \geq 0$ the polynomial identity

$$3\sum_{r=0}^{k-1}(n-1-2r)^5 = k(n-k)\Big(3(n-k)^4 + 10(n-k)^2(k^2-1) + (k^2-1)(3k^2-7)\Big). \tag{4.3}$$

For $k = 0$ both sides vanish, and the inductive step follows from the algebraic identity (verified directly by expansion, and checked symbolically by `ring` in Lean) that replacing k by $k+1$ in the right-hand side of (4.3) adds precisely $3(n-1-2k)^5$. Setting $k = s$ and subtracting $3(n-s)s^5$, we simplify the inner factor:

$$\begin{aligned} 3\,F_{5,n}(\sigma_{n,s}) &= s(n-s)\Big(3(n-s)^4 + 10(n-s)^2(s^2-1) + (s^2-1)(3s^2-7) - 3s^4\Big) \\ &= s(n-s)\Big(3(n-s)^4 + (10s^2-10)(n-s)^2 - 10s^2 + 7\Big) \\ &= s(n-s)\big((n-s)^2-1\big)\big(3(n-s)^2 + 10s^2 - 7\big), \end{aligned}$$

which is (1.4). □

Remark 4.1 (Asymptotic agreement with Pain [11]). Setting $s = \alpha n$ and letting $n \to \infty$, Theorem 1.5 gives $F_{3,n}(\sigma_{n,\alpha n}) \sim \alpha(1-\alpha)^3 n^4$, whose maximum occurs at $\alpha^* = 1/4$ with leading coefficient $27/256 \approx 0.10547$, while Theorem 1.6 gives

$$F_{5,n}(\sigma_{n,\alpha n}) \sim \frac{1}{3}\,\alpha(1-\alpha)^3\big(3(1-\alpha)^2 + 10\alpha^2\big)\,n^6 = \frac{1}{3}\,\alpha(1-\alpha)^3\big(3 - 6\alpha + 13\alpha^2\big)\,n^6,$$

whose maximum occurs at $\alpha^* \approx 0.274$ with leading coefficient ≈ 0.0815. These two limiting ratios agree with Pain's analysis [11, Section 3] and lie below the finite bound $1/3$. Equality $s/n = 1/3$ occurs for $n = 3$ and $q = 3$.

Table 1 lists the exact optimal shifts and maxima for $q = 3$ and $q = 5$ across $1 \leq n \leq 12$, computed from Theorems 1.5 and 1.6.

Remark 4.2 (Quintic values in Pain [11]). In the text of Section 3.2 of [11, p. 8], Pain briefly writes that the optimal parameter for $q = 5$ is $s = 2$ for $6 \leq n \leq 12$. However, Pain's own Table 2 on the same page—in complete agreement with Theorem 1.6 and Table 1 above—correctly records $s_{\mathrm{opt}} = 2$ for $6 \leq n \leq 9$ and $s_{\mathrm{opt}} = 3$ for $10 \leq n \leq 12$, giving $F_{5,10}(\sigma_{10,3}) = 77{,}280$, $F_{5,11}(\sigma_{11,3}) = 138{,}600$, and $F_{5,12}(\sigma_{12,3}) = 234{,}720$ (whereas $s = 2$ gives 75,600, 132,480, and 219,780, respectively).

TABLE 1. Exact optimal shift parameters and maximum displacement sums for $q = 3$ and $q = 5$ ($1 \le n \le 12$), alongside the unconditional upper bound $\lfloor n/3 \rfloor$ of Theorem 1.3.

n	$\lfloor n/3 \rfloor$	Optimal s ($q = 3$)	$\max \lvert F_{3,n} \rvert$	Optimal s ($q = 5$)	$\max \lvert F_{5,n} \rvert$
1	0	$0, 1$	0	$0, 1$	0
2	0	$0, 1, 2$	0	$0, 1, 2$	0
3	1	1	6	1	30
4	1	1	24	1	240
5	1	1	60	1	1,020
6	2	$1, 2$	120	2	3,240
7	2	2	240	2	8,640
8	2	2	420	2	19,740
9	3	2	672	2	40,320
10	3	$2, 3$	1,008	3	77,280
11	3	3	1,512	3	138,600
12	4	3	2,160	3	234,720

5. Formal Verification in Lean 4

The Lean 4 file [5, 6] proves the finite optimality theorem, the odd-power specialization, the cubic maximum and a named maximizing shift, and the quintic evaluation for every admissible shift. The classification of all cubic maximizing shifts follows in the text from (4.2). The asymptotic comparisons and numerical table are evaluations in the manuscript.

- **Proof file:** `proofs/PainOddPowers.lean` (1,238 lines).
- **Toolchain:** Lean `v4.33.1` (`leanprover/lean4:v4.33.1`).
- **Mathlib commit:** `0df444a360eaa60ab8c11dca51a86af692955474`.
- **Replay command:** `lake env lean proofs/PainOddPowers.lean`.
- **Verification status:** Compiles with exit code `0`, zero `sorry`s, and zero warnings.
- **Axiom audit:** Running `#print axioms` on all four main theorems confirms dependence strictly on the three standard Lean axioms `propext`, `Classical.choice`, and `Quot.sound`.

Table 2 identifies the kernel-checked declarations used for the results in namespace `PainOddPowers`.

Table 2: Kernel-checked declarations in `proofs/PainOddPowers.lean`.

Paper result	**Lean declaration (`PainOddPowers.*`)**
Definitions 1.1–1.2	`permSum`, `F_pow`, `IsOddConvex`, `delta`, `shiftRevPerm`, `shiftRevVal`
Lemma 2.1	`IsOddConvex.map_zero`, `sum_delta_eq`, `sum_delta_Ico_eq`, `IsOddConvex.delta_le_of_le`, `IsOddConvex.secant_ineq`
Lemma 2.2	`isOddConvex_pow`
Lemma 2.3	`permSum_shiftRevPerm`, `shiftRevVal_succ_sub`
Proposition 2.4	`shiftRevVal_zero_le_one`, `shiftRevVal_le_pred_of_three_mul_gt`, `exists_opt_s`
Definition 3.1	`dualC`, `dualA`, `dualB`
Proposition 3.2	`dual_params_spec`
Definition 3.3	`dualU`, `dualV`

Paper result	**Lean declaration (`PainOddPowers.*`)**
Lemma 3.4	`sum_two_deltas_eq`, `dual_quad1_eq`, `dual_sum_eq`
Lemma 3.5	`linear_bound_le`
Lemma 3.6	`dual_quad4_feasible`
Proposition 3.7	`dual_feasible`
Theorem 1.3	`max_permSum_oddConvex`
Theorem 1.4	`F_pow_inv_eq_neg`, `pain_conjecture_all_odd_powers`
Cubic value and optimizer	`F_pow_three_shiftRevPerm`, `pain_exact_cube_formula`
Theorem 1.6	`pain_exact_fifth_formula`

Contributions and use of AI

Cole Dexter and Gemini 4 Neon are the coauthors of the original work. Gemini 4 Neon, operating through Antigravity, developed the finite dual argument and range reduction, obtained the cubic and quintic evaluations, wrote and checked the Lean formalization, and drafted the original manuscript. GPT 6.2 Astra, operating through Codex, assisted with the historical source audit, the cubic endpoint and numerical checks, and editorial revisions clarifying the scope of formal verification.

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