MINIMUM ORDERS FOR THREE SMALL TRIANGLE-REGULAR PARAMETER PAIRS

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Abstract. Hak, Kozerenko, Lohvynov and Yarosh singled out four pairs of small vertex and triangle degrees for which they did not know an example. We determine the minimum orders for three of the pairs. They are 15 for $(8, 17)$, 18 for $(9, 17)$ and 18 for $(10, 23)$. We give explicit connected graphs of diameter two. The lower bounds use triangle incidence, complementation, and the impossibility of a cubic graph with two triangles at every vertex. For the latter two pairs we also identify minimum-order witnesses in the 2020 Royle–Holt census. The Caro–Mifsud database also tabulates an order-18 example for $(9, 17)$. Lean verifies the three minimum orders. A separate exhaustive enumeration shows that the order-15 witness for $(8, 17)$ is unique up to isomorphism.

1. Three small parameter pairs

Let G be a finite nonempty simple graph. It is (r, c)-triangle-regular if every vertex has degree r and belongs to exactly c distinct triangles. Equivalently, the open neighborhood of each vertex induces c edges. Caro et al. [1, §5.1] studied the set of achievable c at fixed r. Sheffield and Xi [4] obtained further restrictions on this set for general graphs and for abelian Cayley graphs. For small degrees, Hak, Kozerenko, Lohvynov and Yarosh [5, §5 and Appendix A, Table 1] listed $(8, 17)$, $(9, 17)$, $(9, 23)$ and $(10, 23)$ as the first four pairs for which existence was unknown to them. Their accompanying matrices cover several lower parameter pairs [6]. For the fourth pair, $(9, 23)$, we obtain a lower bound of 18 on the order of any witness. Its existence is an open question in [5, §5].

Theorem 1. *The minimum order of an (r, c)-triangle-regular graph is* 15*,* 18*, and* 18 *for $(r, c) = (8, 17)$, $(9, 17)$, and $(10, 23)$, respectively. The minima are also attained by connected graphs of diameter two.*

The 2020 vertex-transitive graph census of Royle and Holt [8] already contains order-18 witnesses for $(9, 17)$ and $(10, 23)$. We identify their records below. The database of Caro and Mifsud [3] explicitly lists the order-18 graph `NoncayleyTransitive, {18, 3}` with parameters $(9, 17)$. Their paper [2] tabulates small orders through degree six. Our proof of minimality covers arbitrary finite graphs, including those without vertex transitivity. The three certificates tabulated here distinguish vertices by a local invariant, so each is non-vertex-transitive.

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2. Lower bounds from complementation

Let $n = |V(G)|$ and put $d = n - 1 - r$. The handshake and triangle-incidence identities give

$$n > r, \qquad 2 \mid nr, \qquad 3 \mid nc.$$

Indeed, the two products count the incidences of vertices with edges and with triangles. The complement identity below is due to Nair and Vijayakumar [9, Corollary 2.5].

Lemma 2. *If G has parameters (r, c) and order n, its complement has parameters (d, c'), where*

$$c' = \binom{d}{2} - \frac{r(n-2r)}{2} - c.$$

In particular $0 \le c' \le \binom{d}{2}$.

Proof. For a vertex v, let $A = N_G(v)$ and $B = V(G) \setminus (A \cup \{v\})$. The r vertices of A account for r^2 degree incidences. Of these, r go to v and $2c$ remain inside A. There are therefore $r(r-1) - 2c$ edges from A to B. Subtracting these, the r edges at v, and the c edges in A from $nr/2$ leaves $r(n-2r)/2 + c$ edges in B. The complement induces precisely the other pairs in B, its open neighborhood of v. □

Lemma 3. *No nonempty cubic graph has exactly two triangles through each vertex.*

Proof. Caro and Mifsud excluded $(3, 2)$ in [2, §4.1]. It also appears in [5, Appendix A, Table 1]. For a direct proof, the three neighbors of any vertex v must induce a two-edge path, say a, b, c in order. The neighbors of b are then a, c, v, and the neighbors of v are a, b, c. In particular a is adjacent to b, v and not to c. Neither b nor v can be adjacent to the third neighbor of a, since both have their three neighbors already. Thus a belongs only to triangle abv, a contradiction. □

The incidence conditions leave the following orders strictly below the claimed minima. Applying Lemma 2 gives a complete list of the complementary parameters at those orders.

(r, c)	n	d	c'	$\binom{d}{2}$
$(8, 17)$	9	0	11	0
	12	3	2	3
$(9, 17)$	12	2	11	1
$(10, 23)$	12	1	17	0
	15	4	8	6

All but the second line have $c' > \binom{d}{2}$. Lemma 3 excludes the remaining line. This proves the lower bounds in Theorem 1. For $(9, 23)$ the incidence conditions leave only $n = 12$ below 18. Its complementary parameters would be $(2, 5)$, impossible since $\binom{2}{2} = 1$.

3. Finite certificates

Number the vertices from 0 through $n - 1$. In the table below, entry R_i encodes the open neighborhood of vertex i. More precisely, the binary digit of weight 2^j in R_i is one precisely when ij is an edge. All entries are written in decimal. For $(8, 17)$ the complement has a compact description. Label the rim edges of the ten-vertex Möbius ladder $s_i = \{i, i+1\}$ for $i \in \mathbb{Z}/10$, and its rungs $h_j = \{j, j+5\}$ for $0 \le j < 5$. To its line graph add $s_i s_{i+3}$ for all i modulo ten and $h_j h_{j+2}$ for all j modulo five. These two cycles raise the degree to six and introduce no triangles. For instance two rim edges joined by a new edge have no common endpoint and no ladder edge joining their endpoints. The same holds for a joined pair of

rungs. Two steps in either new cycle lead to disjoint ladder edges, and the new cycles have lengths ten and five. For the other two witnesses we used degree-preserving edge switches on graphs of order 18, seeking complements with parameters $(8, 11)$ and $(7, 8)$.

i	$(8,17)$	$(9,17)$	$(10,23)$	i	$(8,17)$	$(9,17)$	$(10,23)$
0	22448	42358	186210	9	20853	40124	93515
1	15720	233593	146161	10	25003	127557	226069
2	27352	155281	183672	11	12510	221852	209694
3	28038	170946	47988	12	2931	206478	117915
4	23237	84775	89294	13	3310	252993	249386
5	14147	181139	237967	14	1949	44468	46997
6	14902	206219	164767	15	–	26411	94445
7	27677	23404	250226	16	–	146770	179888
8	22059	114937	20205	17	–	79982	77031

Proof of the upper bounds in Theorem 1. For each column decode R_i as a set S_i. The diagonal digits are zero and $j \in S_i$ is equivalent to $i \in S_j$. Each S_i has respectively 8, 9, or 10 entries. The number of triangles at i can be recovered by counting each of the neighborhood's edges twice, giving

$$2c_i = \sum_{j \in S_i} |S_i \cap S_j|.$$

The sums are constant in their columns, with values 34, 34 and 46. Any two nonadjacent vertices share a neighbor in each graph. None of the graphs is complete, so their diameters are exactly two. Together these checks give the asserted connected witnesses. □

As a second verification, direct enumeration of all unordered triples in the three matrices gives 85, 102 and 138 triangles. They have 60, 81 and 90 edges, respectively. For each vertex i, form the multiset of $|S_i \cap S_j|$ as j ranges over S_i. The three graphs have respectively 2, 11 and 9 distinct such multisets, proving the asserted failure of vertex transitivity. The Lean certificates define finite simple graphs from the row integers, prove the graph and neighborhood-count statements using kernel reduction, and derive the lower bounds from the incidence and complement identities.

4. Uniqueness at order fifteen

Theorem 4 (Finite enumeration)**.** *Up to isomorphism there is one* $(8, 17)$*-triangle-regular graph on* 15 *vertices.*

Proof. By Lemma 2, the complement H has parameters $(6, 2)$. It has ten triangles because $15 \cdot 2/3 = 10$. Form a multigraph M on these ten triangles. Each point of H lies in two distinct triangles and gives an edge of M between them. Each triangle contains three points, so M is cubic. Two distinct triangles share at most two points, so its edge multiplicities are at most two.

The underlying simple graph B of M is triangle-free. Otherwise, from three pairwise adjacent triangles A, B', C choose a point in each of the intersections $A \cap B'$, $B' \cap C$, $C \cap A$. Those three points are distinct and pairwise adjacent in H. Their new triangle cannot be any of A, B', C, since each point already belongs to its two specified triangles. This contradicts triangle degree two. The degrees in the support are therefore two or three, and the doubled edges form a perfect matching on the vertices of degree two in the support.

Now label each point by its edge of M. Pairs of edges incident at a common vertex of M give forced adjacencies in H. These already contain all ten required triangles, so a

completion to degree six may add only edges that create no further triangle. Conversely, such a completion gives the required complement: the only triangles in its forced adjacency graph are the ten stars of three edges, since the support is triangle-free.

We enumerated triangle-free simple supports on ten vertices with minimum degree two and maximum degree three, including disconnected supports, using `geng` [7]. For each support we generated every perfect matching in the subgraph induced by its degree-two vertices and tried all degree-six completions. There were 167 support classes, 38 matching choices across these representatives, and exactly one completion before any isomorphism reduction. An independent vertex-augmentation enumeration reproduced the supports, and a separate Boolean satisfiability encoding of the allowed extra edges confirmed that there was one completion, isomorphic to the complement of the displayed matrix. The programs and commands accompany the paper. This finite uniqueness enumeration runs separately from the Lean proof of Theorem 1. □

5. Witnesses in the 2020 census

Royle and Holt deposited all vertex-transitive graphs of orders 10 through 47 in September 2020 [8]. In their order-18 archive `alltrans18.tar`, number the graph6 records inside each degree-specific gzip member starting at one. For degree 9, records 35, 36, 44 and 47 of `alltrans18_k=09.gz` each have triangle degree 17. For degree 10, records 31 and 41 of `alltrans18_k=10.gz` each have triangle degree 23. The record numbers can be checked by decoding the rows of the graph6 upper triangle in column order, and then counting the edges induced by the neighbors of each vertex. Thus the existence of graphs realizing those two pairs can also be read directly from the earlier census. The archived proof and certificate bundle includes all three matrices, the Lean theorem combining attainment with minimality, and a separate verifier for the matrices and census records.

References

[1] Y. Caro, J. Lauri, X. Mifsud, R. Yuster and C. Zarb, Flip colouring of graphs, *Graphs Combin.* **40** (2024), article 106. doi:10.1007/s00373-024-02838-w.

[2] Y. Caro and X. Mifsud, On (r, c)-constant, planar and circulant graphs, *Discuss. Math. Graph Theory* **45** (2025), 707–723. doi:10.7151/dmgt.2551.

[3] Y. Caro and X. Mifsud, *The (r, c)-graph database* (2024). General graph table.

[4] N. S. Sheffield and Z. Xi, Graphs with the same edge count in each neighborhood, arXiv:2507.14473 (2025). `https://arxiv.org/abs/2507.14473`.

[5] A. Hak, S. Kozerenko, D. Lohvynov and Y. Yarosh, Regular K_3-regular graphs, arXiv:2602.23517v2 (2026), §5 and Appendix A. `https://arxiv.org/abs/2602.23517`.

[6] D. Lohvynov and collaborators, *Regular K_3-regular graphs: examples*, supplemental adjacency matrices, March 2026. Archived source revision.

[7] B. D. McKay and A. Piperno, Practical graph isomorphism, II, *J. Symbolic Comput.* **60** (2014), 94–112. doi:10.1016/j.jsc.2013.09.003.

[8] G. Royle and D. Holt, *Vertex-transitive graphs on fewer than 48 vertices*, Zenodo, September 2020. doi:10.5281/zenodo.4010122.

[9] B. Radhakrishnan Nair and A. Vijayakumar, About triangles in a graph and its complement, *Discrete Math.* **131** (1994), 205–210. doi:10.1016/0012-365X(94)90385-9.