FORBIDDEN TRIANGLE DEGREES AT DEFICITS OF ORDER $R^{3/2}$

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Abstract. Let $\mathrm{Spec}(r)$ consist of the integers c for which some finite nonempty r-regular graph has exactly c triangles at every vertex. For all sufficiently large r, the smallest $c \in \{0, \dots, \binom{r}{2}\}$ outside $\mathrm{Spec}(r)$ has deficit $\binom{r}{2} - c$ of order $r^{3/2}$, answering a question of Sheffield and Xi for unrestricted graphs. For $r > 0$, if $\delta = \binom{r}{2} - c$ is realized, then 2δ lies at most $128(1 + \lfloor \delta/r \rfloor)^2$ below some multiple of r. When this upper bound is smaller than r, the sharp bound is $0 \le rx - 2\delta \le x(x-1)$ with $x = \lceil 2\delta/r \rceil$. Closed neighborhoods with large overlap form equivalence classes. Counting missing edges in a class shows why the integer multiple arises. Explicit forbidden values interlace with Cartesian-product examples. At the centers of the narrow bands, every connected realization is a balanced complete multipartite graph. For each sufficiently narrow band of index $k \ge 2$, we exclude positive offsets below k and characterize when offset k occurs. For one family in the fifth band we give separate congruence criteria for unrestricted graphs and abelian Cayley graphs. For infinitely many degrees the parameters force every realization to lack vertex transitivity.

1. The unrestricted spectrum

All graphs are finite, simple and nonempty. A graph is (r, c)-triangle-regular when every vertex has degree r and lies in exactly c triangles. Equivalently, every open neighborhood induces c edges. Write $B_r = \binom{r}{2}$ and $\mathrm{Spec}(r) = \{c : \text{an } (r, c)\text{-triangle-regular graph exists}\}$. A connected component preserves both parameters, so restricting existence to connected graphs gives the same set.

Caro, Lauri, Mifsud, Yuster and Zarb proved that every integer $0 \le c \le r^2/2 - 5r^{3/2}$ is realized [2, Theorem 14(i)]. Sheffield and Xi obtained a nonexistence interval immediately below B_r, and narrow bands for abelian Cayley graphs. They asked in [5, §1.1] whether some $c = B_r - \Omega(r^{3/2})$ is missing among arbitrary graphs and whether the remaining interval has a single threshold. More recently, Hak, Kozerenko, Lohvynov and Yarosh obtained exact restrictions near the maximum and characterized several Turán graphs [3]. We give a necessary band condition for arbitrary graphs and deduce missing values at the scale of the known existence bound.

Theorem 1.1. *If $r > 0$ and $c \in \mathrm{Spec}(r)$, put $\delta = B_r - c$. There is an integer $x \ge 0$ satisfying*

$$0 \le rx - 2\delta \le 128\big(1 + \lfloor \delta/r \rfloor\big)^2.$$

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Theorem 1.2. *For every $r \geq 4096$, put*

$$t = \left\lfloor \frac{\sqrt{r}}{64} \right\rfloor, \qquad D = \left\lfloor \frac{r(2t+1)}{4} \right\rfloor.$$

Then $B_r - D \notin \operatorname{Spec}(r)$, and $r\lfloor\sqrt{r}\rfloor/256 \leq D \leq B_r$. In particular, if $c_{\min}(r) = \min(\{0, \dots, B_r\} \setminus \operatorname{Spec}(r))$, then

$$B_r - c_{\min}(r) = \Theta(r^{3/2}).$$

The doubled deficit makes the band condition integral for both parities of r. The proof uses neither a symmetry assumption nor a bound on the number of vertices.

2. Closed neighborhoods and their classes

Put $s = r + 1$ and $S_v = N[v]$. Each row S_v has s elements. For rows set $d(u, v) = |S_u \setminus S_v| = s - |S_u \cap S_v|$. This is half of the symmetric-difference distance, and it satisfies the triangle inequality. Every missing edge in S_v has both ends in $N(v)$, so there are exactly δ such edges. Consequently

$$\sum_{z \in S_v} d(v, z) = 2\delta. \tag{1}$$

Indeed, $d(v, z) = |S_v \setminus S_z|$ counts the nonneighbors of z in S_v. Ben-Or, Coppersmith, Luby and Rubinfeld [1] used an overlap-distance argument for translates of a group subset. We apply it to the rows of an arbitrary symmetric incidence relation. Sheffield and Xi derived their abelian-group estimate by Fourier analysis [5, §4.2].

Lemma 2.1. *Suppose $100\delta \leq s^2$ and set $A = \lfloor 4\delta/s \rfloor$. The relation $u \sim v$ defined by $|S_u \cap S_v| > s/2$ is an equivalence relation. If H is the class of v, then both $|S_v \setminus H|$ and $|H \setminus S_v|$ are at most A.*

Proof. For $p = |S_u \cap S_v|$, sum the triangle inequality $d(u, v) \leq d(u, z) + d(z, v)$ over the common vertices z. Equation (1) gives $p(s - p) \leq 4\delta$. If $p > s/2$, then $s - p \leq 8\delta/s$. Two successive related pairs therefore have endpoint distance at most $16\delta/s < s/2$. This proves transitivity. Reflexivity and symmetry follow from the definition.

Each $z \in S_v \setminus H$ contributes at least $s/2$ to (1). Thus $|S_v \setminus H| \leq 4\delta/s$. To bound $H \setminus S_v$, count pairs (z, w) with $z \in S_v$, $w \in S_z \setminus S_v$. By symmetry their number is 2δ. Each $w \in H \setminus S_v$ accounts for more than $s/2$ of the pairs because $|S_w \cap S_v| > s/2$. The bounds are integer-valued, so the floor is valid. □

3. Counting the missing edges

For a vertex set U write $m(U)$ for its number of unordered pairs of distinct nonadjacent vertices. Fix a class H, put $q = |H|$ and $M = m(H)$. For $v \in H$ let

$$X = S_v \cap H, \qquad Y = S_v \setminus H, \qquad Z = H \setminus S_v, \qquad a = |Y|, \quad f = |Z|.$$

Then $q + a = s + f$ and $a, f \leq A$. Every $y \notin H$ has at most A neighbors in H, since these are outside its own equivalence class.

Lemma 3.1. *For every $v \in H$,*

$$0 \leq sa + M - \delta \leq 3A^2.$$

If $3A^2 < s$, then a, f are constant on H, and $2M = qf$.

Proof. The number of missing cross pairs from X to Y is $(s-a)a-e(X,Y)$, with $e(X,Y) \le aA$. Expanding $m(S_v) = \delta$ gives

$$sa + M - \delta = a^2 + e(X,Y) - m(Y) + M - m(X).$$

Here $m(Y) \le \binom{a}{2} \le a^2$. Every missing pair counted in $M - m(X)$ has an end in Z, and each such end misses at most A vertices in H. Hence $0 \le M - m(X) \le fA$. These estimates prove the first assertion.

The terms M and δ do not depend on v. If two outside counts a in the same class differed, their multiples by s would differ by at least s. They all lie in one interval of length $3A^2$. Thus a is constant, and so is $f = q+a-s$. Each vertex of H misses f others there. Counting their incidences gives $2M = qf$. □

Proof of Theorem 1.1. Let $L = 1+\lfloor \delta/r \rfloor$. If $r \le 100L^2$, take $x = \lceil 2\delta/r \rceil$, so $0 \le rx-2\delta < r$. Otherwise $\delta < rL$, $100\delta < s^2$ and $A \le 4L$. Also $3A^2 \le 48L^2 < s$. Lemmas 2.1 and 3.1 apply. Write $E = sa + M - \delta$, so $0 \le E \le 3A^2$, and take $x = 2a + f$. Substitution gives

$$2\delta - rx = x + (f-a)f - 2E.$$

Since $x \le 3A$ and $|(f-a)f| \le A^2$, the absolute error is at most $3A + 7A^2 \le 124L^2$. To determine its sign, put $F = H \setminus S_v$ and let $e(F)$ denote the number of edges induced by F. Each vertex of F misses f vertices of H. Removing F therefore removes $f^2-m(F)$ missing pairs, counting those internal to F just once. This gives $m(X) = qf/2 - f^2 + m(F)$. The vertex $v \in X$ is adjacent to every vertex of Y, so there are at most $(s-a)a - a$ missing cross pairs. Bounding $m(Y)$ by $\binom{a}{2}$ and substituting $q = s + f - a$ gives

$$2\delta \le r(2a+f) - a(a+f+1) - 2e(F). \tag{2}$$

In particular $2\delta \le rx$. □

Theorem 3.2. *Put $L = 1 + \lfloor \delta/r \rfloor$. If $128L^2 < r$ and δ is realized, then $x = \lceil 2\delta/r \rceil$ satisfies*

$$0 \le rx - 2\delta \le x(x-1).$$

For every fixed $x \ge 1$, each endpoint occurs for infinitely many degrees.

Proof. For each class, the integer $2a+f$ places $r(2a+f)$ in the interval $[2\delta, 2\delta+128L^2]$. Its length is less than r, so every class gives the same integer x. In particular, every exterior vertex has at most $x/2$ neighbors outside its own class, and hence in H. It follows that $2e(X,Y) \le ax$. The exact missing-pair count used above can be written as

$$rx - 2\delta = af + 2a^2 - 2a + 2e(F) + 2e(X,Y) - 2m(Y).$$

Use $2e(F) \le f(f-1)$ and $m(Y) \ge 0$. The upper bound becomes $2af+4a^2-2a+f(f-1)$, at most $(2a+f)(2a+f-1) = x(x-1)$.

For the endpoint $rx - 2\delta = 0$, take the complement of m disjoint copies of K_{x+1}. Its degree is $(m-1)(x+1)$ and its deficit is $(m-1)\binom{x+1}{2}$. For the other endpoint use the complement of m disjoint copies of $K_{x,x}$. Its degree is $2mx - x - 1$. In the open neighborhood of a vertex, each of the other $m-1$ copies contributes x^2 missing edges, and none comes from its own copy. Thus $\delta = (m-1)x^2$, giving $rx - 2\delta = x(x-1)$. Both families satisfy $128L^2 < r$ for all sufficiently large m. □

4. Missing values and realized neighbors

Proof of Theorem 1.2. Write $z = \lfloor \sqrt{r} \rfloor$. Then $t = \lfloor z/64 \rfloor \geq 1$ and $32(t+1) \leq 64t \leq z < 64(t+1) \leq 128t$. As $4D \leq r(2t+1) \leq 4D+3$, we have $rt \leq 2D$, which yields $rz \leq 256D$. Also $D \leq rt \leq B_r$, since $2t \leq r-1$. The band half-width at D is at most $128(t+1)^2 \leq r/8$. But $2D$ differs from the midpoint $r(t+1/2)$ by at most $3/2$, so it is at distance at least $r/2 - 3/2$ from any integer multiple of r. Theorem 1.1 rules out $B_r - D$.

The deficit of the smallest missing c is at least D. On the other hand, the existence theorem of Caro et al. [2, Theorem 14(i)] makes that deficit strictly less than $5r^{3/2} - r/2$. The two estimates give the stated order. □

Corollary 4.1. *If $k \geq 0$ and $4096(k+1)^2 \leq r$, there is a forbidden deficit strictly between the two realized deficits $k(r-k)$ and $(k+1)(r-k-1)$.*

Proof. The Cartesian product $K_{r-k+1} \,\square\, K_{k+1}$ is r-regular. The neighbors reached along each factor span a clique, and the two groups have no edges between them. Thus $c = \binom{r-k}{2} + \binom{k}{2}$ and the deficit is $k(r-k)$. The same argument with $k+1$ gives the other endpoint. These are the two-factor instances of the construction in [2, Theorem 14(i)].

Take $D_k = rk + \lfloor r/4 \rfloor$. It exceeds the first endpoint, and the difference from it to the second is $r - (k+1)^2 - \lfloor r/4 \rfloor > 0$. Here $\lfloor D_k/r \rfloor = k$, so the bound in Theorem 1.1 is $128(k+1)^2 \leq r/32$. The distance of $2D_k$ from $r\mathbb{Z}$ is at least $r/2 - 3/2$. This excludes D_k. □

As k varies there are $\lfloor \sqrt{r}/64 \rfloor$ successive intervals of this kind. Thus the unrestricted spectrum has repeated gaps even within a distance of order $r^{3/2}$ from the top. Sheffield and Xi's separate Conjecture 1 asked whether every realized pair has an abelian Cayley realization. The trureturing formalization [6] shows that the icosahedron realizes $(5, 5)$ while no abelian Cayley graph realizes this pair.

5. Rigidity at a band center

Hak et al. [3, Question 3] ask for degree and triangle-degree characterizations of Turán graphs. In the following family the graph order is unrestricted. The two local degrees determine the order of each component as well as its isomorphism type. An earlier theorem of Šoltés [4, Theorem 1] recognizes connected regular graphs as balanced complete multipartite graphs when their open neighborhoods are regular of the same, sufficiently large degree. That condition fixes the triangle count on each edge. Here only the total count at each vertex is prescribed. Write $K_{t,\dots,t}$ for the complete multipartite graph with t vertices in each part.

Theorem 5.1. *Let $k \geq 0$ be an integer. Suppose a graph is (r, c)-triangle-regular with $2(B_r - c) = rk$ and $r > 128(k+1)^2$. Every connected component is complete multipartite with all parts of size $k+1$. A graph with these parameters exists exactly when $k+1$ divides r. In particular, for $m > 128t + 1$, the degree and triangle degree of $K_{t,\dots,t}$ with m parts determine it uniquely among connected graphs.*

Proof. Here $L = 1 + \lfloor \delta/r \rfloor \leq k+1$, so we are in the second case of Theorem 1.1 in every class H. The integer $x = 2a + f$ obtained there satisfies $|rk - rx| < r$. Hence $x = k$.

Apply (2) with $F = H \setminus S_v$. Since $2\delta = rk$ and $2a + f = k$, it gives $a = 0$, $f = k$ and $e(F) = 0$ for every vertex v.

There are no edges between distinct classes. Each class is connected, since two vertices of the class share a vertex of their closed neighborhoods. Within a class the complement graph is k-regular, and the neighbors of each vertex form a clique because $e(F) = 0$. On any shortest path of length at least two in a component of the complement, the first three vertices contradict shortestness. Thus its components are cliques of size $k+1$, which are the parts of the asserted complete multipartite graph.

With m parts of size $t = k+1$, a vertex has degree $(m-1)t$. Its neighbors occupy $m-1$ whole parts, so the number of missing pairs in its neighborhood is $(m-1)\binom{t}{2} = rk/2$. This gives the divisibility criterion and the converse construction. If $m > 128t+1$, then $(m-1)t > 128t^2$, as required. □

The scale of the recognition bound. The quadratic dependence on the part size is necessary up to a constant factor. Šoltés [4, p. 216] credits Nedela with the equal-factor categorical product example. For even $b \geq 4$, take instead $K_{b+1} \times K_b$, with two ordered pairs adjacent precisely when both coordinates differ. Any two vertices have a common neighbor, so the graph is connected. Its degree is $r = b(b-1)$ and each adjacent pair has $(b-1)(b-2)$ common neighbors. Therefore $2c = r(b-1)(b-2)$ and $2\delta = r(2b-3)$. Set $t = 2b-2$ and $m = 1 + b/2$. The complete m-partite graph with part size t has exactly these local parameters, yet the product has $b(b+1) = mt+2$ vertices. Here $r/t^2 \to 1/4$. Thus degree and local triangle count can fail to determine the component even at degrees of order t^2.

A rectangular rook graph gives the exact ratio $r/t^2 = 1/2$. For $s \geq 2$, the connected graphs $K_{s+1} \,\square\, K_{s(2s-1)+1}$ and $K_{2s,\ldots,2s}$ with $s+1$ parts have degree $r = 2s^2$ and triangle degree $\binom{s}{2}(2s)^2$. Their orders differ by $(s+1)(2s-1)(s-1) > 0$. Taking $t = 2s$ gives the asserted ratio.

6. The first attainable offset inside a band

Fix $k \geq 2$ and put $\eta = rk - 2(B_r - c)$. The center criterion of Theorem 5.1 concerns $\eta = 0$. We next characterize when the smallest possible positive offset, k, is attained.

Theorem 6.1. *Suppose $r > 128(k+1)^2$ and $0 < \eta \leq k$. Then $c \in \mathrm{Spec}(r)$ if and only if $\eta = k$ and*

$$k-1 \mid r-1 \qquad \text{or} \qquad \big(2 \mid k \text{ and } k+2 \mid r-1\big).$$

Furthermore, $c \notin \mathrm{Spec}(r)$ whenever $k < \eta < 2(k-1)$.

Proof. Write $S_v = N[v]$. For either interval in the statement, $\eta < r$ and $1 + \lfloor \delta/r \rfloor \leq k+1$. The proof of Theorem 1.1 assigns every class H an index $2a+f$ with $2\delta \leq r(2a+f) \leq 2\delta + 128(k+1)^2$. The only possible index is k. Equation (2) gives

$$\eta \geq a(a+f+1) + 2e(H \setminus S_v).$$

If $a \geq 2$, then $a(a+f+1) \geq 2(k-1)$, since the difference is $(a-2)(a+f-1)$. For $k = 2$, the equation $2a+f = 2$ already gives $a \leq 1$. Thus all classes have $a = 0$ or 1. Every edge between classes is the only exterior edge at both ends, and consequently belongs to no triangle.

We use the complement identity of Nair and Vijayakumar [7, Corollary 2.5]. Let D be d-regular on $q = d+f+1$ vertices and $J = \overline{D}$. If D and J have local triangle deficits $\delta_D(v)$ and $\delta_J(v)$, then

$$2\delta_D(v) + 2\delta_J(v) = df. \tag{3}$$

Indeed, J is f-regular. Its edges inside $N_D[v]$ account for $\delta_D(v)$. Its edges inside $F = N_J(v)$ number $\binom{f}{2} - \delta_J(v)$. Subtracting from the $qf/2$ edges of J all edges with at least one end in F gives $\delta_D(v) = qf/2 - f^2 + \binom{f}{2} - \delta_J(v)$.

Apply this identity to $D = G[H]$. The degree and the deficit of J are $(k, \eta/2)$ for $a = 0$, and $(k-2, (\eta-k)/2)$ for $a = 1$. For the second calculation, D has degree $r-1$ and has the same triangle count at each vertex as G, since the exterior edge lies in no triangle.

We recall the low-deficit facts needed for J. In degree d, a constant doubled deficit t with $0 < t \le d$ must equal d. At $t = d$ with $d \ge 3$, d is even and every component is a complete graph on $d+2$ vertices with a perfect matching removed. There is also no value $d < t < 2(d-1)$. These statements follow from [3, Corollaries 3.7–3.8 and Theorem 4.1]; the cases $d \le 2$ follow by examining paths and cycles. The kernel proof obtains them directly from

$$|S_u \cap S_v| \, |S_u \setminus S_v| \le t.$$

To see this inequality, put $C = S_u \cap S_v$. Each $x \in C$ misses at least $|S_u \setminus S_v|$ members of $S_u \cup S_v$. Every resulting ordered nonedge is covered by the sum of the two unordered nonedge counts in S_u, S_v: a nonedge within C occurs once in each count, and a nonedge leaving C occurs in only one of them. Their sum is t.

For $a = 0$, a positive $\eta \le k$ therefore forces $\eta = k$. If $k \ge 3$, then k is even and $k+2$ divides $|H| = r+k+1$, hence divides $r-1$. When $k = 2$, the first divisor in the theorem is 1. For $a = 1$, the complement deficit is nonnegative, so $\eta \le k$ again forces $\eta = k$. At equality its components are cliques of size $k-1$: every open neighborhood is a clique, which precludes a shortest path of length two in a component. Thus $k-1$ divides $|H| = r+k-2$, and divides $r-1$. If $k < \eta < 2(k-1)$, the $a = 0$ case violates the second low-deficit gap above. The $a = 1$ case would give a positive doubled deficit strictly below $k-2$ in degree $k-2$. This proves both nonexistence assertions.

For the converse, suppose $k-1 \mid r-1$. Take two copies of the complete multipartite graph with $1 + (r-1)/(k-1)$ parts of size $k-1$ and match corresponding vertices across them. The matching contributes one to each degree and creates no triangles. The doubled deficit is $(r-1)(k-2) + 2(r-1) = k(r-1)$. Finally, suppose k is even and $k+2 \mid r-1$. Partition $r+k+1$ vertices into blocks of size $k+2$. On each block take a complete graph with a perfect matching removed, and complement their disjoint union. The union is k-regular with doubled deficit k at each vertex. Equation (3) gives doubled deficit $rk-k$ in the complement, as required. □

Corollary 6.2. *For $r > 2048$ the entire band with index 3 has the following spectrum. Put $\eta = 3r - 2(B_r - c)$ and suppose $0 \le \eta \le 6$. Then*

η	0	3	4	6
$c \in \mathrm{Spec}(r)$ *exactly when*	$4 \mid r$	$2 \mid r-1$	$6 \mid r+4$	$2 \mid r$.

The offsets $1, 2, 5$ do not occur.

Proof. Theorems 5.1 and 6.1 handle $\eta \le 3$. Throughout the band $2a + f = 3$, so again $a \le 1$. For $a = 1$ the complement in the class has degree one and therefore zero deficit, forcing $\eta = 3$. For $a = 0$ the class has $r+4$ vertices and its complement is cubic with $3 - \eta/2$ triangles through each vertex. Thus η is even. When $\eta = 4$, each vertex belongs to a unique triangle, so the order is divisible by three. The order is even by the handshaking lemma. Conversely, on any multiple of six vertices take disjoint triangular prisms. Each is cubic with one triangle through each vertex, and complementing this union gives $\eta = 4$

by (3). At $\eta = 6$ the complement must be cubic and triangle-free, so its order is even. For every even $q = r + 4$ in the stated range, the prism $C_{q/2} \,\square\, K_2$ is cubic and triangle-free; its complement realizes the required pair. □

7. A residue obstruction to Cayley realization

Sheffield and Xi asked whether every realizable pair (r, c) has an abelian Cayley realization [5, Conjecture 1]. The Omega Institute's trureturing project gave a counterexample at $(5, 5)$ in November 2026 [6]. Its graph witness is the icosahedron, already recorded by Caro and Mifsud [8] and Hak et al. [3, Appendix A]. We distinguish the two spectra at infinitely many degrees, within a linear deficit of the maximum. A subfamily excludes Cayley graphs of all finite groups.

Theorem 7.1. *Suppose $r \geq 1200$ and $2\binom{r}{2} - c) = 5r - 16$. Then*

1. *An (r, c)-triangle-regular graph exists if and only if $6 \mid r + 6$.*
2. *A finite abelian Cayley realization exists if and only if $18 \mid r + 6$.*
3. *If $r + 6 = 6p$ with p prime, some finite group has a Cayley graph realizing (r, c) if and only if $p \equiv 1 \pmod 3$.*

Proof. Write $s = r + 1$, $\delta = \binom{r}{2} - c$, and $S_v = N[v]$. We first determine the order of the component containing v. The high-overlap classes from the proof of the band theorem apply because $100\delta \leq s^2$ and $3\lfloor 4\delta/s \rfloor^2 < s$. Indeed $4\delta/s < 10$. In each class H write $a = |S_v \setminus H|$ and $f = |H \setminus S_v|$. The local block estimate gives

$$\left| r(2a + f) - 2\delta \right| \leq 128 \cdot 3^2.$$

Here $1 + \lfloor \delta/r \rfloor = 3$. Since $2\delta = 5r - 16$ and $r \geq 1200$, the only possible index is $2a + f = 5$. In particular every vertex has at most two neighbors outside its class.

Put $X = S_v \cap H$, $Y = S_v \setminus H$, $F = H \setminus S_v$. Write $m(U)$ for missing unordered pairs within U, and $e(X, Y)$ for the number of edges crossing disjoint sets. Every vertex of H misses exactly f vertices there, so

$$m(X) = |H| f/2 - f^2 + m(F).$$

Using $|H| = s + f - a$, $|X| = s - a$, and $m(Y) = \binom{a}{2} - e(Y)$ in the count of missing pairs of S_v gives

$$5r - 2\delta = f^2 + af + a^2 - a - f - 2m(F) + 2e(X, Y) + 2e(Y).$$

Each vertex of Y belongs to a different class from H and has at most two neighbors in X. Thus the right side is at most $f^2 + af + 2a^2 + 2a - f$. This bound equals 13 for $(a, f) = (1, 3)$ and 14 for $(2, 1)$. The required value is 16, so $a = 0$ and $f = 5$. The class has $q = r + 6$ vertices. No edges leave it, and any two of its vertices share a closed neighbor, so it is a connected component.

Let J be the complement inside the component. Its degree is five. The complement identity for a regular graph and its complement states that their doubled local triangle deficits sum to the product of their degrees. Consequently

$$2\left(\binom{5}{2} - \text{triangles of } J \text{ at } v\right) = 5r - 2\delta = 16.$$

Hence every vertex of J lies in two triangles. Conversely, complementing any $(5, 2)$-triangle-regular graph on $q = r + 6$ vertices gives the required pair. The edge and triangle incidence sums of J give $2 \mid 5q$ and $3 \mid 2q$, hence $6 \mid q$.

There are $(5,2)$-triangle-regular graphs on both 12 and 18 vertices. For order 12, take the line graph of the three-dimensional cube and match each edge of the cube to its antipodal edge. Each original line-graph vertex has degree four and lies in the two triangles formed by the stars at its endpoints. Antipodal cube edges have no common neighbor in the line graph, so the matching adds no triangles. At order 18 take the Cartesian product $K_3 \,\square\, K_3 \,\square\, K_2$. The coordinate degrees are $2, 2, 1$ and the first two factors each supply one triangle at a vertex. Every multiple $q = 6j$ with $j \geq 2$ is a sum of 12's and 18's: use only 12's when j is even and one 18 when j is odd. Disjoint union and complementation prove the first assertion.

Now suppose the dense graph is a Cayley graph. Left translations preserve the high-overlap equivalence relation, so its class at the identity is a subgroup H. Indeed $1 \sim x$ implies $x^{-1} \sim 1$, and $1 \sim x$, $1 \sim y$ imply $1 \sim x \sim xy$. Restricting to H and complementing gives another Cayley graph, of order q, now with local parameters $(5,2)$.

For an inverse-closed connection set D omitting the identity in any group, the oriented edges $u \to v$ in the identity neighborhood correspond bijectively to triples

$$(u, u^{-1}v, v^{-1}) \in D^3, \qquad abc = 1.$$

There are four such triples. Cyclic rotation preserves their product-one condition. The number of fixed triples is therefore congruent to four modulo three. The fixed triples are (a, a, a) with $a^3 = 1$, and inversion pairs their nonidentity entries without fixed points. Thus all four triples are fixed and H contains at least four distinct nonidentity elements of order three. In an abelian group these include two distinct order-three subgroups, whose product has order nine. This gives $18 \mid q$. Conversely, for $q = 18j$, use the group $C_3 \times C_3 \times C_2 \times C_j$ with the two opposite standard generators in each of the first two coordinates and the nonzero generator in the third. Its Cayley graph consists of j copies of the order-18 witness above, and its complement is another abelian Cayley graph. This proves the second assertion.

Finally let $q = 6p$ as in the theorem. Then $p > 6$. First suppose $p \equiv 2 \pmod 3$. By Sylow's theorems [9, §4.5], the Sylow p-subgroup is normal, since its number divides six and is congruent to one modulo p. The quotient of order six has a unique Sylow 3-subgroup. Its inverse image in H has order $3p$ and contains every element of order three. Within that inverse image the number of Sylow 3-subgroups divides p and is congruent to one modulo three. As $p \equiv 2 \pmod 3$, the number is one. This leaves only two nonidentity elements of order three, a contradiction.

For $p \equiv 1 \pmod 3$, choose ω of order three in $\mathbb{F}_p^{\times}$. The affine maps $x \mapsto \omega^i x + b$, $i \in \{0, 1, 2\}$ and $b \in \mathbb{F}_p$, form a group P of order $3p$. The stabilizers U, V of 0 and 1 have order three and intersect trivially, since a nonidentity affine map cannot fix both points. In $P \times C_2$ take the four nonidentity elements of $U \times \{0\}$ and $V \times \{0\}$, together with the generator of the C_2 factor. A product-one triple cannot use the latter generator: it would have to occur twice, forcing the remaining generator to be the identity. Of three generators from U and V, at least two lie in the same subgroup. Cyclic rotation puts this pair first, forcing the third into that subgroup as well. An order-three subgroup gives precisely the two constant triples on its nonidentity elements. The connection set therefore gives exactly four ordered triples, hence two triangles at every vertex. Its Cayley complement on $6p$ vertices realizes the claimed dense pair. □

The unrestricted and abelian verdicts thus differ for the two residue classes $r \equiv 0, 6 \pmod{18}$. There are arbitrarily large primes $p \equiv 2 \pmod 3$: if their list were finite, three

times their product minus one would have a prime factor congruent to two modulo three outside that list. Taking $r = 6p - 6$ gives arbitrarily large graphic pairs with no Cayley realization even over a nonabelian group.

Corollary 7.2. *Suppose $r \geq 1200$, $2(\binom{r}{2} - c) = 5r - 16$, and $r + 6 = 6p$ for a prime $p \equiv 2 \pmod 3$. There is no vertex-transitive (r, c)-triangle-regular graph. Thus for arbitrarily large degrees some realizable local parameters admit no vertex-transitive realization.*

Proof. Suppose a vertex-transitive realization exists. The high-overlap relation is invariant under automorphisms, so the setwise stabilizer of each class is transitive within it: an automorphism taking one class member to another must preserve their class. By Theorem 7.1 and its block argument, complementing the induced graph on one class yields a vertex-transitive graph J on $q = 6p$ vertices, each of degree five and on two triangles.

By vertex transitivity, either the two triangles through each vertex share an edge at every vertex or they do so at none. If the triangles share an edge at every vertex, the edges on two triangles form a perfect matching. Counting incidences of these edges with triangles gives q incidences. Yet J has only $2q/3$ triangles, each containing at most one matching edge. Therefore distinct triangles of J share at most one vertex.

Define D to have the triangles of J as vertices. Each vertex v of J specifies an edge of D joining the two triangles through v. These edges are distinct, so D is simple. Each triangle contains three vertices and therefore has degree three in D. Moreover D is triangle-free: three pairwise meeting triangles of J would have three distinct intersection vertices, which would themselves span a further triangle of J through each intersection vertex. The action on $V(J)$ makes D edge-transitive. Its $4p$ vertices split into isomorphic connected components, each itself edge-transitive. A component has even order by cubicity and at least six vertices by simplicity and triangle-freeness. Its order therefore equals $2p$ or $4p$.

For order $2p$, Folkman's theorem [10] makes a regular edge-transitive graph vertex-transitive. Cheng and Oxley's classification [11, Theorem 2.4] restricts a connected cubic example to $p = 2, 3, 5$ or $p \equiv 1 \pmod 3$. For order $4p$, Alaeiyan and Onagh [12, Corollary 3.2] exclude semisymmetric cubic graphs. A vertex- and edge-transitive graph of odd degree is arc-transitive [11, Proposition 1.2], so Feng and Kwak's list [13, Theorem 6.2] applies. Its orders of the form $4p$ are 8, 20 and 28. Both cases contradict $p \geq 11$ and $p \equiv 2 \pmod 3$. The final assertion follows from the ordinary realizations in Theorem 7.1 and the infinitude of these primes. □

For example, $p = 227$ gives $r = 1356$ and $c = 915308$. An ordinary graph realizes these local parameters, and the corollary excludes every vertex-transitive realization.

Formal verification

The Lean development models G as a finite simple graph. A double count identifies edges in the induced neighborhood with three-vertex cliques through the center. The kernel checks both one-sided band estimates, the explicit missing parameter of Theorem 1.2, the forbidden deficits and both product witnesses in Corollary 4.1, and the parameters of the canonical balanced graphs. It also verifies an isomorphism from each component at a band center to a complete equipartite graph. The two graph families attaining the endpoints of Theorem 3.2 are also checked for arbitrary numbers of copies. Lean verifies both graph parameters, connectivity, and nonisomorphism for the rectangular rook example. The kernel also checks the first-offset criterion, all cases in the third band, and the

fifth-band criteria for unrestricted and abelian Cayley graphs. When $r + 6 = 6p$ with p prime, it also checks the criterion for Cayley realization over arbitrary finite groups. The vertex-transitivity obstruction in Corollary 7.2 is a written deduction using the cited classifications of cubic graphs. The unequal-factor categorical product is counted above. The upper estimate in the asymptotic conclusion invokes the published existence theorem [2]. The proof source and replay script and supporting modules give the exact statements.

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