THE MINIMUM ORDER OF TWO-COLOUR FLIP GRAPHS

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ABSTRACT. In a two-colour flip graph each vertex has blue degree b and red degree $r > b$, yet every induced closed neighbourhood contains more blue edges than red edges. Caro, Lauri, Mifsud, Yuster and Zarb characterized the realizable degree pairs. We determine the order of magnitude of their minimum realization order $h(b, r)$ throughout this range. With $s = b+1$ and $\delta = \binom{s}{2} - 1 - r$, we prove that $h(b, r)$ lies between $rs^2/(s+2\delta)$ and 48 times this expression. Along any sequence with $b \to \infty$, we have $h(b, r) = \Theta(b+r)$ if and only if $\delta = \Omega(b^2)$. At the maximum red degree ($\delta = 0$) we determine the exact minimum $s(r + 1)$ and classify the minimizing graphs. For the smallest admissible pair we prove that $h(3, 4) = 14$, improving the previously published example of order 16. The proofs use a triangle-count defect, a second-moment bound on overlapping blue stars and a two-sided multipartite construction.

1. Minimum realization order

All graphs are finite, simple, undirected and nonempty. For a graph whose edges are coloured blue or red, let $N_B(v)$ and $N_R(v)$ be the sets of blue and red neighbours of v, respectively. Write $N[v] = \{v\} \cup N_B(v) \cup N_R(v)$ for its closed neighbourhood. A (b, r)-*flip graph* has $|N_B(v)| = b$ and $|N_R(v)| = r$ for every v, with $b < r$. Moreover, the subgraph induced by each $N[v]$ has strictly more blue edges than red edges.

Caro, Lauri, Mifsud, Yuster and Zarb proved that such a graph exists exactly when

$$3 \le b < r \le \binom{b+1}{2} - 1. \tag{1}$$

Their coloured Cartesian-product construction gives the general bound $2r(b + 1)$, which they also sharpened for suitable parameters. They asked for the minimum realization order $h(b, r)$ or a nontrivial lower bound [1, Sections 1 and 7]. Mifsud gave constructions on $O(b + r)$ vertices for $4 \le b < r < b + 2\lfloor (b + 2)/6 \rfloor^2$ and asked where the linear-order regime ends [2, Problem 2]. Sheffield and Xi studied the related problem of prescribing a common neighbourhood edge count and obtained further results on flip sequences with more colours [3]. We give a uniform estimate for all pairs in (1). Throughout let

$$s = b + 1, \qquad \delta = \binom{s}{2} - 1 - r, \qquad L(b, r) = \frac{rs^2}{s + 2\delta}.$$

Theorem 1.1. *For every admissible pair,*

$$L(b, r) \le h(b, r) \le 48L(b, r).$$

At the maximum red degree $r = \binom{b+1}{2} - 1$*, the exact value is*

$$h(b, r) = (b + 1)(r + 1) = (b + 1)\binom{b+1}{2}.$$

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The lower estimate will come from bounding the average overlap of the closed blue stars centred at the red neighbours of a vertex. To match it, we build blue graphs with dense neighbourhoods in each fibre, then couple fibres using a regular bipartite red graph. The exact equality case follows from additional structure at $\delta = 0$. We also prove that $h(3,4) = 14$ (Theorem 2.3). Figure 1 of [1] and Figure 1.1 of [2] give a 16-vertex example and describe it as the smallest known. The exact order for general interior pairs remains open.

Corollary 1.2. *For any family of admissible pairs with $b \to \infty$, the estimate $h(b,r) = \Theta(b+r)$ holds with uniform constants if and only if $\delta = \Omega(b^2)$ along the family. If $\delta = o(b^2)$ then $h(b,r)/(b+r) \to \infty$. If $\delta = o(b)$ then $h(b,r) \sim r(b+1)$.*

Proof. Since $b < r$, the factor $r/(b+r)$ lies between $1/2$ and 1. Thus Theorem 1.1 makes $h(b,r)/(b+r)$ comparable, with absolute constants, to $s^2/(s+2\delta)$. Also $h(b,r) \geq b+r+1$ by the total degree. These facts give the first two assertions. For the last, compare the lower bound with the upper bound $s(r+1)$ constructed in Proposition 3.1. □

2. A defect identity and blue-star overlaps

Write $D(v)$ for the number of blue edges minus the number of red edges in $G[N[v]]$. Let T_i count triangles containing exactly i blue edges. Let W count the open blue wedges, that is, choices of a vertex and an unordered pair of its blue neighbours whose endpoints are nonadjacent in G. Put $n = |V(G)|$.

Lemma 2.1. *Every (b,r)-flip graph satisfies*

$$\sum_v (D(v) - 1) + W + T_1 + 3T_0 = n\delta. \tag{2}$$

Proof. An edge incident to a centre v contributes to $D(v)$ according to its own colour. The remaining contributions come from edges opposite v in a triangle. Summing over the three possible centres of each triangle gives

$$\sum_v D(v) = n(b-r) + 3T_3 + T_2 - T_1 - 3T_0.$$

Counting pairs of blue neighbours instead gives $n\binom{b}{2} = 3T_3 + T_2 + W$. Subtraction yields (2). This retains the slack terms in the triangle count used for the degree restriction in [1]. □

For $y \in V(G)$ put $S_y = \{y\} \cup N_B(y)$, so $|S_y| = s$. For $v \in V(G)$ define

$$O_v = \sum_{\{y,z\} \subset N_R(v)} |S_y \cap S_z|.$$

All sums over pairs in this paper use distinct unordered pairs.

Lemma 2.2. *For some vertex v we have $O_v \leq r\delta$.*

Proof. Sum O_v over v and first consider adjacent endpoints y, z. The triples $(v, \{y,z\})$ are closed red wedges, numbering $T_1 + 3T_0$. Each star intersection has size at most s. If y, z are nonadjacent, every member of $S_y \cap S_z$ is a common blue neighbour. Summed over nonadjacent endpoint pairs these common neighbours give precisely W. The pair y, z has at most r common red neighbours that could have served as v. Consequently

$$\sum_v O_v \leq s(T_1 + 3T_0) + rW \leq rn\delta,$$

using $r \geq s$ and Lemma 2.1, since $D(v) \geq 1$. Averaging proves the claim. □

Choose v as in Lemma 2.2. For each vertex w, let X_w count the stars S_y, $y \in N_R(v)$, that contain w. Then

$$\sum_w X_w = rs, \qquad \sum_w X_w^2 = rs + 2O_v \le r(s + 2\delta).$$

Cauchy–Schwarz gives $(rs)^2 \le nr(s + 2\delta)$, proving the lower bound in Theorem 1.1. For the integer refinement, observe that $X_v = 0$. Indeed v is distinct from each red neighbour y, and the red edge vy precludes a blue edge between them. For each integer $j \ge 0$, sum $(X_w - j)(X_w - j - 1) \ge 0$ over $w \ne v$. There are $n - 1$ summands, and the first two moments are unchanged by omitting v. This gives

$$2jrs \le j(j + 1)(n - 1) + 2r\delta. \tag{3}$$

In particular, the choice $j = 1$ gives $n \ge 1 + r(s - \delta)$.

Theorem 2.3. *The smallest* $(3, 4)$*-flip graph has* 14 *vertices.*

Proof. For $(b, r) = (3, 4)$ we have $s = 4$ and $\delta = 1$. The bound after (3) gives $n \ge 13$. The blue graph is cubic, so $3n$ is even and $n \ge 14$.

For the matching construction, take the dihedral group of order 14 with rotation a and reflection u, satisfying $a^7 = u^2 = 1$ and $uau = a^{-1}$. Join g to gh in blue or red according to the two generator sets

$$B = \{u, a, a^{-1}\}, \qquad R = \{au, a^2, a^4u, a^{-2}\}.$$

The sets are disjoint, inverse-closed and omit the identity, giving a simple graph with the required degrees. Left translations preserve both colours, so it suffices to examine $N[1] = \{1\} \cup B \cup R$. Besides the three blue and four red edges incident to 1, the edges of this neighbourhood are

$$\begin{aligned} &\text{blue:} \quad \{u, au\},\ \{a, au\},\ \{a, a^2\},\ \{a^{-2}, a^{-1}\},\\ &\text{red:} \quad \{u, a^{-1}\},\ \{a, a^{-1}\}. \end{aligned}$$

These follow by testing $g^{-1}g'$ against B and R. The induced counts are seven blue and six red at every vertex, as required. □

3. The maximum red degree

Proposition 3.1. *For every pair in* (1)*,* $h(b, r) \le s(r + 1)$*.*

Proof. Take vertices (i, a) with $0 \le i \le r$ and $a \in \mathbb{Z}/s\mathbb{Z}$. Each fibre of fixed i is a blue clique. For $i < j$ add the red perfect matching $(i, a)(j, a + 1)$. Each vertex has the two required degrees. A red triangle in fibres $i < j < k$ would require $a + 2 = a + 1$ modulo s, which is impossible. A triangle with exactly two red edges would have its blue edge inside one fibre and contradict the matching property between two fibres. Thus each closed neighbourhood contains r red edges, all incident to its centre. Its blue edges are precisely the $\binom{s}{2}$ edges of its central fibre. The required flip follows. □

Lemma 3.2. *If* $\delta = 0$*, every blue connected component is a blue* K_s*, or a* K_{s+1} *whose blue edges omit a perfect matching restored by red edges. Between any two components the red edges form a matching.*

Proof. At $\delta = 0$ all terms in (2) vanish. Thus $W = 0$ and no triangle has two or more red edges. The blue neighbours of a vertex induce a complete graph in the two-colour graph. Inside any such clique the red edges form a matching.

Suppose a, u, v, d is a shortest blue path of length three. Choose a third blue neighbour t of u distinct from a, v, possible since $b \ge 3$. The vertices a, v are nonadjacent in blue, so they are matched red partners inside $N_B(u)$. Hence t is joined in blue to both a and v. Inside $N_B(v)$ the red partners u, d force the third vertex t to be joined in blue to d. The resulting blue path a, t, d

is shorter, a contradiction. Any longer shortest path contains a shortest three-edge subpath. Each blue component therefore has diameter at most two. Since $W = 0$, two vertices at blue distance two are adjacent in the two-colour graph. Thus each component induces a complete two-colour graph.

Red edges inside this complete graph form a matching. Blue regularity forces each vertex to have the same internal red degree, either zero or one. These two cases give respectively the two components in the statement. Finally, two red neighbours of a vertex cannot lie in the same other component, as they would form a triangle with two red edges. □

Theorem 3.3. *At $\delta = 0$ the minimum graphs consist of $r + 1$ disjoint blue K_s's, one perfect red matching between every pair of blue cliques, and no all-red triangle. Conversely every such matching system is a minimum graph.*

Proof. If some blue component is a K_s, one of its vertices requires r red neighbours in different other components by Lemma 3.2. So there are at least $r + 1$ components of size at least s. If all components are of the second type, each vertex needs $r - 1$ external red neighbours in distinct components, giving at least $r(s + 1) > s(r + 1)$ vertices. Here $r > s$ for $b \geq 3$ at the boundary.

Equality is possible by Proposition 3.1. It forces exactly $r + 1$ components, all blue K_s's. Red regularity then gives perfect matchings between every pair. Conversely, for any such system with no red triangle, the same neighbourhood count as in Proposition 3.1 gives $\binom{s}{2} = r + 1$ blue edges and r red edges at every vertex. This proves both the exact formula and the classification. □

4. A multipartite blue layer and a bipartite red layer

We now construct smaller graphs when the slack δ allows. The multipartite blue fibres supply blue edges near each vertex. To obtain red neighbours we use vertices in parts with the same label on the opposite side.

Lemma 4.1. *Suppose $m, k \geq 1$, $t \geq 4$ is even, $0 \leq d = (t - 1)m - b < 2m$, and $km \geq r$. There is a blue/red graph on $2ktm$ vertices with degrees (b, r) such that each closed neighbourhood contains exactly r red edges and at least*

$$\binom{s}{2} - \frac{b(m - 1 + d)}{2}$$

blue edges.

Proof. Take two sides, each comprising k fibres. In each fibre start with a complete t-partite graph with m vertices in each part. To find d disjoint perfect matchings, split the t part labels into two equal halves. All edges between these halves are present. Each half has $tm/2 \geq 2m$ vertices. Number the two halves modulo $tm/2$. Between them the matchings $a \mapsto a + j$ for $0 \leq j < d$ are pairwise edge-disjoint and span the fibre.

Delete these d matchings and colour the remaining edges blue. Every vertex has blue degree b. Relative to the other vertices in its fibre it has exactly $m - 1 + d$ blue nonneighbours. Hence among the b blue neighbours of any vertex, at most $b(m - 1 + d)/2$ unordered pairs lack a blue edge. Including the b incident blue edges gives the asserted lower bound.

For each part label, the km vertices of that label on the left and the km on the right support a simple r-regular bipartite graph. Number both sets modulo km and take r distinct cyclic shifts as red matchings. A red edge crosses sides and preserves the part label. A blue edge stays on its side and changes the label. A triangle cannot have an odd number of red edges by the side partition, and two consecutive red edges have endpoints of the same label and therefore cannot be closed by a blue edge. Thus every triangle is entirely blue. In particular, every red edge in a closed neighbourhood is incident to its centre. □

Proof of the uniform upper bound in Theorem 1.1. When $\delta < 2b$, Proposition 3.1 suffices, since

$$\frac{s(r+1)}{L(b,r)} = \left(1+\frac{1}{r}\right)\left(1+\frac{2\delta}{s}\right) < 7.$$

Suppose $\delta \geq 2b$. Set $m = \lfloor \delta/(2b) \rfloor$, let t be the smallest even integer with $(t-1)m \geq b$, and set $k = \lfloor r/m \rfloor + 1$. Since $\delta \geq 2b$ and $\delta < b^2/2$ by (1), we have

$$1 \leq m \leq b/4, \qquad \delta \leq 4bm, \qquad 0 \leq d < 2m. \tag{4}$$

In particular $t \geq 4$. The blue edge loss in Lemma 4.1 is less than $3bm/2 \leq 3\delta/4$. Thus the local blue count exceeds r, giving a flip graph of order $n = 2ktm$.

We have $tm = b + m + d \leq b + 3m$ and $km \leq r + m$. Since $m \leq b/4 \leq r/4$, these inequalities give

$$nm = 2(km)(tm) \leq 2(r+m)(b+3m) \leq \tfrac{35}{8} rb.$$

Using $3(s+2\delta) \leq 8\delta$, which holds because $b \geq 3$ and $\delta \geq 2b$, then using (4), we obtain

$$3n(s+2\delta) \leq 8n\delta \leq 32bnm \leq 140rb^2 \leq 144rs^2.$$

Dividing by $3(s+2\delta)$ proves the result. □

Formal verification. The accompanying Lean 4 development checks the graph definitions, the two bounds, the linear-order criterion of Corollary 1.2, the exact minimum and classification at the boundary, and the fourteen-vertex minimum of Theorem 2.3 [4]. It represents the number of edges of an induced subgraph by its number of ordered incidences divided by two and proves agreement with the strict comparison in the definition above. Natural-number inequalities are stated after clearing positive denominators. Instructions in the archive identify the exact environment and the theorem corresponding to each result.

References

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